

# **Finite Difference Methods for Ordinary and Partial Differential Equations**

**Steady-State and Time-Dependent Problems**

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