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978-0-521-89485-2 - Conceptual Mathematics, 2nd Edition: A First Introduction to Categories

F. William Lawvere and Stephen H. Schanuel

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