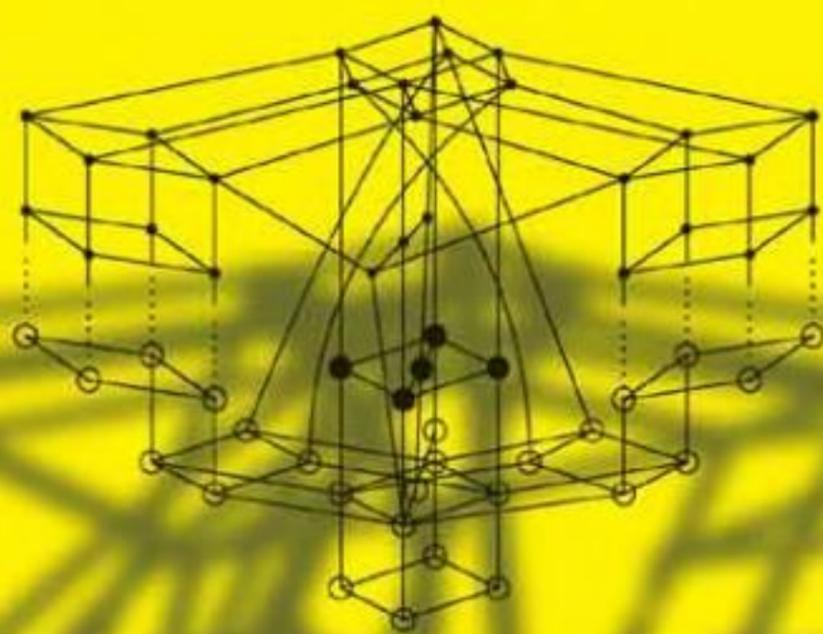


DIETLINDE LAU

# Function Algebras on Finite Sets

A Basic Course on Many-Valued Logic  
and Clone Theory



 Springer

Springer Monographs in Mathematics

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