

DIETLINDE LAU

Function Algebras on Finite Sets

A Basic Course on Many-Valued Logic
and Clone Theory



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