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# Fractal Geometry, Complex Dimensions and Zeta Functions

Geometry and Spectra of Fractal Strings

With 53 Illustrations

 Springer

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The front cover shows a tubular neighborhood of the Devil's staircase (Figure 12.2, page 335) and the quasiperiodic pattern of the complex dimensions of a nonlattice self-similar string (Figure 3.7, page 86).

Mathematics Subject Classification (2000): Primary—11M26, 11M41, 28A75, 28A80, 35P20, 58G25  
Secondary—11J70, 11M06, 11N05, 28A12, 30D35, 81Q20

Library of Congress Control Number: 2006929212

ISBN-10: 0-387-33285-5      e-ISBN: 0-387-35208-2  
ISBN-13: 978-0-387-33285-7

Printed on acid-free paper.

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Printed in the United States of America.      (EB)

9 8 7 6 5 4 3 2 1

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