

# **Grundlehren der mathematischen Wissenschaften 302**

*A Series of Comprehensive Studies in Mathematics*

## *Editors*

M. Artin S. S. Chern J. Coates J. M. Fröhlich  
H. Hironaka F. Hirzebruch L. Hörmander  
C. C. Moore J. K. Moser M. Nagata W. Schmidt  
D. S. Scott Ya. G. Sinai J. Tits M. Waldschmidt  
S. Watanabe

## *Managing Editors*

M. Berger B. Eckmann S. R. S. Varadhan

Herbert Lange   Christina Birkenhake

# Complex Abelian Varieties



Springer-Verlag Berlin Heidelberg GmbH

Herbert Lange  
Christina Birkenhake  
Mathematisches Institut  
Universität Erlangen-Nürnberg  
Bismarckstr. 1 1/2  
W-8520 Erlangen, FRG

Mathematics Subject Classification (1980): 14-02, 14KXX, 30F10,  
32G20

ISBN 978-3-662-02790-5  
DOI 10.1007/978-3-662-02788-2

ISBN 978-3-662-02788-2 (eBook)

Library of Congress Cataloging-in-Publication Data  
Lange, H. (Herbert), 1943- Complex Abelian varieties / Herbert Lange, Christina  
Birkenhake. p. cm. -- (Grundlehren der mathematischen Wissenschaften ; 302)  
Includes bibliographical references and index.

1. Abelian Varieties. 2. Riemann surfaces. I. Birkenhake, Christina. II. Title.  
III. Series. QA564.L32 1992 516.3'53--dc20 92-23806

This work is subject to copyright. All rights are reserved, whether the whole or part  
of the material is concerned, specifically the rights of translation, reprinting, reuse of  
illustrations, recitation, broadcasting, reproduction on microfilm or in any other way,  
and storage in data banks. Duplication of this publication or parts thereof is permitted  
only under the provisions of the German Copyright Law of September 9, 1965, in its  
current version, and permission for use must always be obtained from Springer-  
Verlag. Violations are liable for prosecution under the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1992

Originally published by Springer-Verlag Berlin Heidelberg New York in 1992

Softcover reprint of the hardcover 1st edition 1992

Typesetting: Camera ready by authors  
41/3140-543210 - Printed on acid-free paper

# Contents

<b>Introduction</b>	<b>1</b>
<b>Notation</b>	<b>5</b>
<b>Chapter 1</b>	
<b>Complex Tori</b>	<b>6</b>
§ 1 Complex Tori	7
§ 2 Homomorphisms	9
§ 3 Cohomology of Complex Tori	13
§ 4 The Hodge Decomposition	15
Exercises	20
<b>Chapter 2</b>	
<b>Line Bundles on Complex Tori</b>	<b>23</b>
§ 1 Line Bundles on Complex Tori	24
§ 2 The Appell-Humbert Theorem	29
§ 3 Canonical Factors	32
§ 4 The Dual Complex Torus	35
§ 5 The Poincaré Bundle	39
Exercises	42
<b>Chapter 3</b>	
<b>Cohomology of Line Bundles</b>	<b>46</b>
§ 1 Characteristics	47
§ 2 Theta Functions	51
§ 3 The Positive Semidefinite Case	57
§ 4 The Vanishing Theorem	58
§ 5 Cohomology of Line Bundles	63
§ 6 The Riemann-Roch Theorem	66
Exercises	69
<b>Chapter 4</b>	
<b>Abelian Varieties</b>	<b>71</b>
§ 1 Polarized Abelian Varieties	72
§ 2 The Riemann Relations	75
§ 3 The Decomposition Theorem	77
§ 4 The Gauss Map	82
§ 5 Projective Embeddings	85
§ 6 Symmetric Line Bundles	89
§ 7 Symmetric Divisors	93
§ 8 Kummer Varieties	98
§ 9 Morphisms into Abelian Varieties	101
§ 10 The Pontryagin Product	104
§ 11 Homological Versus Numerical Equivalence	107
Exercises	111

**Chapter 5**

<b>Endomorphisms of Abelian Varieties</b>	<b>115</b>
§ 1 The Rosati Involution	116
§ 2 Polarizations	121
§ 3 Norm-Endomorphisms and Symmetric Idempotents	125
§ 4 Endomorphisms Associated to Cycles	129
§ 5 The Endomorphism Algebra of a Simple Abelian Variety	133
Exercises	142

**Chapter 6**

<b>Theta and Heisenberg Groups</b>	<b>147</b>
§ 1 Theta Groups	148
§ 2 Theta Groups under Homomorphisms	152
§ 3 The Commutator Map	154
§ 4 The Canonical Representation of the Theta-Group	156
§ 5 The Isogeny Theorem	159
§ 6 Heisenberg Groups and Theta Structures	162
§ 7 The Schrödinger Representation	167
§ 8 The Isogeny Theorem for Finite Theta Functions	170
§ 9 Symmetric Theta Structures	173
Exercises	178

**Chapter 7**

<b>Equations for Abelian Varieties</b>	<b>182</b>
§ 1 The Multiplication Formula	183
§ 2 Surjectivity of the Multiplication Map	187
§ 3 Projective Normality	190
§ 4 The Ideal of an Abelian Variety in $\mathbb{P}_N$	194
§ 5 Riemann's Theta Relations	200
§ 6 Cubic Theta Relations	202
Exercises	208

**Chapter 8**

<b>Moduli</b>	<b>212</b>
§ 1 The Siegel Upper Half Space	213
§ 2 The Analytic Moduli Space	216
§ 3 Level Structures	220
§ 4 The Theta Transformation Formula, Preliminary Version	224
§ 5 Classical Theta Functions	227
§ 6 The Theta Transformation Formula, Final Version	231
§ 7 The Universal Family	233
§ 8 The Action of the Symplectic Group	236
§ 9 Orthogonal Level $D$ -Structures	238
§ 10 The Embedding of $\mathcal{A}_D(D)_0$ into Projective Space	240
Exercises	244

<b>Chapter 9</b>	
<b>Moduli Spaces of Abelian Varieties with Endomorphism Structure</b>	<b>247</b>
§ 1 Abelian Varieties with Endomorphism Structure	249
§ 2 Abelian Varieties with Real Multiplication	250
§ 3 Some Notation	256
§ 4 Families of Abelian Varieties with Totally Indefinite Quaternion Multiplication	259
§ 5 Families of Abelian Varieties with Totally Definite Quaternion Multiplication	262
§ 6 Families of Abelian Varieties with Complex Multiplication	268
§ 7 Group Actions on $\mathcal{H}_{r,s}$ and $\mathcal{H}_m$	273
§ 8 Shimura Varieties	276
§ 9 The Endomorphism Algebra of a General Member	280
Exercises	286
<b>Chapter 10</b>	
<b>Abelian Surfaces</b>	<b>288</b>
§ 1 Preliminaries	289
§ 2 The 16 <sub>6</sub> -Configuration of the Kummer Surface	292
§ 3 An Equation for the Kummer Surface	297
§ 4 Reider's Theorem	301
§ 5 Polarizations of Type (1, 4)	308
§ 6 Products of Elliptic Curves	313
Exercises	317
<b>Chapter 11</b>	
<b>Jacobian Varieties</b>	<b>320</b>
§ 1 Definition of the Jacobian Variety	321
§ 2 The Theta Divisor	327
§ 3 The Poincaré Bundles for a Curve $C$	333
§ 4 The Universal Property	336
§ 5 Correspondences	340
§ 6 Endomorphisms Associated to Curves and Divisors	342
§ 7 Examples of Jacobians	344
§ 8 The Criterion of Matsusaka-Ran	348
§ 9 Trisecants of the Kummer Variety	352
§ 10 Fay's Trisecant Identity	355
Exercises	361
<b>Chapter 12</b>	
<b>Prym Varieties</b>	<b>365</b>
§ 1 Abelian Subvarieties of a Principally Polarized Abelian Variety	367
§ 2 Prym-Tyurin Varieties	371
§ 3 Prym Varieties	375
§ 4 Topological Construction of Prym Varieties	378
§ 5 The Abel-Prym Map	381

§ 6 The Theta Divisor of a Prym Variety	385
§ 7 Recilla's Theorem	390
§ 8 Donagi's Tetragonal Construction	393
§ 9 Kanev's Criterion	399
Exercises	405
<b>Appendix</b>	
A Algebraic Varieties and Complex Analytic Spaces	409
B Line Bundles and Factors of Automorphy	412
<b>Bibliography</b>	417
<b>Glossary of Notation</b>	427
<b>Index</b>	431