A Brief Table of Integrals

1. $\int u\,dv = uv - \int v\,du$

2. $\int a^u\,du = \frac{a^u}{\ln a} + C, \qquad a \neq 1, \qquad a > 0$

3. $\int \cos u\,du = \sin u + C$

4. $\int \sin u\,du = -\cos u + C$

5. $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, \qquad n \neq -1$

6. $\int (ax+b)^{-1}\,dx = \frac{1}{a}\ln|ax+b| + C$

7. $\int x(ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a^2}\left[\frac{ax+b}{n+2} - \frac{b}{n+1}\right] + C, \qquad n \neq -1, -2$

8. $\int x(ax+b)^{-1}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C$

9. $\int x(ax+b)^{-2}\,dx = \frac{1}{a^2}\left[\ln|ax+b| + \frac{b}{ax+b}\right] + C$

10. $\int \frac{dx}{x(ax+b)} = \frac{1}{b}\ln\left|\frac{x}{ax+b}\right| + C$

11. $\int (\sqrt{ax+b})^n\,dx = \frac{2}{a}\,\frac{(\sqrt{ax+b})^{n+2}}{n+2} + C, \qquad n \neq -2$

12. $\int \frac{\sqrt{ax+b}}{x}\,dx = 2\sqrt{ax+b} + b\int \frac{dx}{x\sqrt{ax+b}}$

13. (a) $\int \frac{dx}{x\sqrt{ax+b}} = \frac{2}{\sqrt{-b}}\tan^{-1}\sqrt{\frac{ax+b}{-b}} + C, \qquad \text{if} \quad b < 0$

(b) $\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C, \qquad \text{if} \quad b > 0$

14. $\int \frac{\sqrt{ax+b}}{x^2}\,dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{ax+b}} + C$

15. $\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}} + C$

16. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

17. $\int \frac{dx}{(a^2+x^2)^2} = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + C$

18. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$

19. $\int \frac{dx}{(a^2-x^2)^2} = \frac{x}{2a^2(a^2-x^2)} + \frac{1}{2a^2}\int \frac{dx}{a^2-x^2}$

20. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a} + C = \ln|x+\sqrt{a^2+x^2}| + C$

21. $\displaystyle\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + C$

22. $\displaystyle\int x^2\sqrt{a^2+x^2}\,dx = \frac{x(a^2+2x^2)\sqrt{a^2+x^2}}{8} - \frac{a^4}{8}\sinh^{-1}\frac{x}{a} + C$

23. $\displaystyle\int \frac{\sqrt{a^2+x^2}}{x}\,dx = \sqrt{a^2+x^2} - a\sinh^{-1}\left|\frac{a}{x}\right| + C$

24. $\displaystyle\int \frac{\sqrt{a^2+x^2}}{x^2}\,dx = \sinh^{-1}\frac{x}{a} - \frac{\sqrt{a^2+x^2}}{x} + C$

25. $\displaystyle\int \frac{x^2}{\sqrt{a^2+x^2}}\,dx = -\frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{x\sqrt{a^2+x^2}}{2} + C$

26. $\displaystyle\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right| + C$

27. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{a^2x} + C$

28. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

29. $\displaystyle\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

30. $\displaystyle\int x^2\sqrt{a^2-x^2}\,dx = \frac{a^4}{8}\sin^{-1}\frac{x}{a} - \frac{1}{8}x\sqrt{a^2-x^2}\,(a^2-2x^2) + C$

31. $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$

32. $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\sin^{-1}\frac{x}{a} - \frac{\sqrt{a^2-x^2}}{x} + C$

33. $\displaystyle\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{1}{2}x\sqrt{a^2-x^2} + C$

34. $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$

35. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C$

36. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C = \ln\left|x+\sqrt{x^2-a^2}\right| + C$

37. $\displaystyle\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + C$

38. $\displaystyle\int (\sqrt{x^2-a^2})^n\,dx = \frac{x(\sqrt{x^2-a^2})^n}{n+1} - \frac{na^2}{n+1}\int (\sqrt{x^2-a^2})^{n-2}\,dx, \qquad n \neq -1$

39. $\displaystyle\int \frac{dx}{(\sqrt{x^2-a^2})^n} = \frac{x(\sqrt{x^2-a^2})^{2-n}}{(2-n)a^2} - \frac{n-3}{(n-2)a^2}\int \frac{dx}{(\sqrt{x^2-a^2})^{n-2}}, \qquad n \neq 2$

40. $\displaystyle\int x(\sqrt{x^2-a^2})^n\,dx = \frac{(\sqrt{x^2-a^2})^{n+2}}{n+2} + C, \qquad n \neq -2$

41. $\displaystyle\int x^2\sqrt{x^2-a^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{x^2-a^2} - \frac{a^4}{8}\cosh^{-1}\frac{x}{a} + C$

42. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\sec^{-1}\left|\frac{x}{a}\right| + C$

43. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^2}\,dx = \cosh^{-1}\frac{x}{a} - \frac{\sqrt{x^2-a^2}}{x} + C$

Continued overleaf.

44. $\displaystyle\int \frac{x^2}{\sqrt{x^2-a^2}}\,dx = \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{x^2-a^2} + C$

45. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C = \frac{1}{a}\cos^{-1}\left|\frac{a}{x}\right| + C$

46. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x} + C$

47. $\displaystyle\int \frac{dx}{\sqrt{2ax-x^2}} = \sin^{-1}\left(\frac{x-a}{a}\right) + C$

48. $\displaystyle\int \sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$

49. $\displaystyle\int (\sqrt{2ax-x^2})^n\,dx = \frac{(x-a)(\sqrt{2ax-x^2})^n}{n+1} + \frac{na^2}{n+1}\int (\sqrt{2ax-x^2})^{n-2}\,dx,$

50. $\displaystyle\int \frac{dx}{(\sqrt{2ax-x^2})^n} = \frac{(x-a)(\sqrt{2ax-x^2})^{2-n}}{(n-2)a^2} + \frac{(n-3)}{(n-2)a^2}\int \frac{dx}{(\sqrt{2ax-x^2})^{n-2}}$

51. $\displaystyle\int x\sqrt{2ax-x^2}\,dx = \frac{(x+a)(2x-3a)\sqrt{2ax-x^2}}{6} + \frac{a^3}{2}\sin^{-1}\frac{x-a}{a} + C$

52. $\displaystyle\int \frac{\sqrt{2ax-x^2}}{x}\,dx = \sqrt{2ax-x^2} + a\sin^{-1}\frac{x-a}{a} + C$

53. $\displaystyle\int \frac{\sqrt{2ax-x^2}}{x^2}\,dx = -2\sqrt{\frac{2a-x}{x}} - \sin^{-1}\left(\frac{x-a}{a}\right) + C$

54. $\displaystyle\int \frac{x\,dx}{\sqrt{2ax-x^2}} = a\sin^{-1}\frac{x-a}{a} - \sqrt{2ax-x^2} + C$

55. $\displaystyle\int \frac{dx}{x\sqrt{2ax-x^2}} = -\frac{1}{a}\sqrt{\frac{2a-x}{x}} + C$

56. $\displaystyle\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$

57. $\displaystyle\int \cos ax\,dx = \frac{1}{a}\sin ax + C$

58. $\displaystyle\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$

59. $\displaystyle\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C$

60. $\displaystyle\int \sin^n ax\,dx = \frac{-\sin^{n-1} ax\cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2} ax\,dx$

61. $\displaystyle\int \cos^n ax\,dx = \frac{\cos^{n-1} ax\sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax\,dx$

62. (a) $\displaystyle\int \sin ax\cos bx\,dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} + C, \qquad a^2 \neq b^2$

(b) $\displaystyle\int \sin ax\sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}, \qquad a^2 \neq b^2$

(c) $\displaystyle\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}, \qquad a^2 \neq b^2$

63. $\displaystyle\int \sin ax\cos ax\,dx = -\frac{\cos 2ax}{4a} + C$

64. $\displaystyle\int \sin^n ax\cos ax\,dx = \frac{\sin^{n+1} ax}{(n+1)a} + C, \qquad n \neq -1$

This table is continued on the endpapers at the back.

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THE FILE • CHALLENGES

Serge Lang

A First Course in Calculus

Fifth Edition

With 367 Illustrations

Springer

Serge Lang
Department of Mathematics
Yale University
New Haven, CT 06520
USA

Mathematics Subjects Classifications (2000): 26-01, 26A06

Library of Congress Cataloging-in-Publication Data
Lang, Serge, 1927–
A first course in calculus.
(Undergraduate texts in mathematics)
Includes index.
1. Calculus. I. Title. II. Series.
QA303.L26 1986 515 85-17181

Printed on acid-free paper.

Previous editions of this book were published in 1978, 1973, 1968, 1964 by Addison-Wesley, Publishing Company, Inc.

Typeset by Composition House Ltd., Salisbury, England.

9 8

ISBN 978-1-4612-6428-6 ISBN 978-1-4419-8532-3 (eBook)
DOI 10.1007/978-1-4419-8532-3 SPIN 10965351

Foreword

The purpose of a first course in calculus is to teach the student the basic notions of derivative and integral, and the basic techniques and applications which accompany them. The very talented students, with an obvious aptitude for mathematics, will rapidly require a course in functions of one real variable, more or less as it is understood by professional mathematicians. This book is not primarily addressed to them (although I hope they will be able to acquire from it a good introduction at an early age).

I have not written this course in the style I would use for an advanced monograph, on sophisticated topics. One writes an advanced monograph for oneself, because one wants to give permanent form to one's vision of some beautiful part of mathematics, not otherwise accessible, somewhat in the manner of a composer setting down his symphony in musical notation.

This book is written for the students to give them an immediate, and pleasant, access to the subject. I hope that I have struck a proper compromise, between dwelling too much on special details and not giving enough technical exercises, necessary to acquire the desired familiarity with the subject. In any case, certain routine habits of sophisticated mathematicians are unsuitable for a first course.

Rigor. This does not mean that so-called rigor has to be abandoned. The logical development of the mathematics of this course from the most basic axioms proceeds through the following stages:

Set theory	Numbers (i.e. real numbers)
Integers (whole numbers)	Limits
Rational numbers (fractions)	Derivatives and forward.

No one in his right mind suggests that one should begin a course with set theory. It happens that the most satisfactory place to jump into the subject is between limits and derivatives. In other words, any student is ready to accept as intuitively obvious the notions of numbers and limits and their basic properties. Experience shows that the students do *not* have the proper psychological background to accept a theoretical study of limits, and resist it tremendously.

In fact, it turns out that one can have the best of both ideas. The arguments which show how the properties of limits can be reduced to those of numbers form a self-contained whole. Logically, it belongs *before* the subject matter of our course. Nevertheless, we have inserted it as an appendix. If some students feel the need for it, they need but read it and visualize it as Chapter 0. In that case, everything that follows is as rigorous as any mathematician would wish it (so far as objects which receive an analytic definition are concerned). Not one word need be changed in any proof. I hope this takes care once and for all of possible controversies concerning so-called rigor.

Most students will not feel any need for it. My opinion is that epsilon-delta should be entirely left out of ordinary calculus classes.

Language and logic. It is not generally recognized that some of the major difficulties in teaching mathematics are analogous to those in teaching a foreign language. (The secondary schools are responsible for this. Proper training in the secondary schools could entirely eliminate this difficulty.) Consequently, I have made great efforts to carry the student verbally, so to say, in using proper mathematical language. It seems to me essential that students be required to write their mathematics papers in full and coherent sentences. A large portion of their difficulties with mathematics stems from their slapping down mathematical symbols and formulas isolated from a meaningful sentence and appropriate quantifiers. Papers should also be required to be neat and legible. They should not look as if a stoned fly had just crawled out of an inkwell. Insisting on reasonable standards of expression will result in drastic improvements of mathematical performance. The systematic use of words like "let," "there exists," "for all," "if... then," "therefore" should be taught, as in sentences like:

Let $f(x)$ be the function such that....
There exists a number such that....
For all numbers x with $0 < x < 1$, we have....
If f is a differentiable function and K a constant such that $f'(x) = Kf(x)$, then $f(x) = Ce^{Kx}$ for some constant C.

Plugging in. I believe that it is unsound to view "theory" as adversary to applications or "computations." The present book treats both as complementary to each other. Almost always a theorem gives a tool for more efficient computations (e.g. Taylor's formula, for computing values of functions). Different classes will of course put different emphasis on them, omitting some proofs, but I have found that if no excessive pedantry is introduced, students are willing, and even eager, to understand the reasons for the truth of a result, i.e. its proof.

It is a disservice to students to teach calculus (or other mathematics, for that matter) in an exclusive framework of "plugging in" ready-made formulas. Proper teaching consists in making the student adept at handling a large number of techniques in a routine manner (in particular, knowing how to plug in), but it also consists in training students in knowing some general principles which will allow them to deal with new situations for which there are no known formulas to plug in.

It is impossible in one semester, or one year, to find the time to deal with all desirable applications (economics, statistics, biology, chemistry, physics, etc.). On the other hand, covering the proper balance between selected applications and selected general principles will equip students to deal with other applications or situations by themselves.

Worked-out problems and exercises. For the convenience of both students and instructors, a large number of worked-out problems has been added in the present edition. Many of these have been put in the answer section, to be referred to as needed. I did this for at least two reasons. First, in the text, they might obscure the main ideas of the course. Second, it is a good idea to make students think about a problem before they see it worked out. They are then much more receptive, and will retain the methods better for having encountered the difficulties (whatever they are, depending on individual students) by themselves. Both the inclusion of worked-out examples and their placement in the answer section was requested by students. Unfortunately, the requirements for good teaching, testing, and academic pressures are in conflict here. The *de facto* tendency is for students to object to being asked to think (even if they fail), because they are afraid of being penalized with bad grades for homework. Instructors may either make too strong requirements on students, or may take the path of least resistance and never require anything beyond plugging in new numbers in a type of exercise which has already been worked out (in class or in the book). I believe that testing conditions (limited time, pressures of other courses and examinations) make it difficult (if not unreasonable) to *test* students other than with basic, routine problems. I do not conclude that the course should consist only of this type of material. Some students often take the attitude that if something is not on tests, then why should it be covered in the course? I object very much to this attitude. I have no global solution to these conflicting pressures.

General organization. I have made no great innovations in the exposition of calculus. Since the subject was discovered some 300 years ago, such innovations were out of the question.

I have cut down the amount of analytic geometry to what is both necessary and sufficient for a general first course in this type of mathematics. For some applications, more is required, but these applications are fairly specialized. For instance, if one needs the special properties concerning the focus of a parabola in a course on optics, then that is the place to present them, not in a general course which is to serve mathematicians, physicists, chemists, biologists, and engineers, to mention but a few. I regard the tremendous emphasis on the analytic geometry of conics which has been the fashion for many years as an unfortunate historical accident. What is important is that the basic idea of representing a graph by a figure in the plane should be thoroughly understood, together with basic examples. The more abstruse properties of ellipses, parabolas, and hyperbolas should be skipped.

Differentiation and the elementary functions are covered first. Integration is covered second. Each makes up a coherent whole. For instance, in the part on differentiation, rate problems occur three times, illustrating the same general principle but in the contexts of several elementary functions (polynomials at first, then trigonometric functions, then inverse functions). This repetition at brief intervals is pedagogically sound, and contributes to the coherence of the subject. It is also natural to slide from integration into Taylor's formula, proved with remainder term by integrating by parts. It would be slightly disagreeable to break this sequence.

Experience has shown that Chapters III through VIII make up an appropriate curriculum for **one term (differentiation and elementary functions)** while Chapters IX through XIII make up an appropriate curriculum for a **second term (integration and Taylor's formula)**. The first two chapters may be used for a quick review by classes which are not especially well prepared.

I find that all these factors more than offset the possible disadvantage that for other courses (physics, chemistry perhaps) integration is needed early. This may be true, but so are the other topics, and unfortunately the course has to be projected in a totally ordered way on the time axis.

In addition to this, studying the log and exponential before integration has the advantage that we meet in a special concrete case the situation where we find an antiderivative by means of area: $\log x$ is the area under $1/x$ between 1 and x. We also see in this concrete case how $dA(x)/dx = f(x)$, where $A(x)$ is the area. This is then done again in full generality when studying the integral. Furthermore, inequalities involving lower sums and upper sums, having already been used in this concrete case, become more easily understandable in the general case. Classes which start the term on integration without having gone through the part on differentiation might well start with the last section of the chapter on logarithms, i.e. the last section of Chapter VIII.

Taylor's formula is proved with the integral form of the remainder, which is then properly estimated. The proof with integration by parts is more natural than the other (differentiating some complicated expression pulled out of nowhere), and is the one which generalizes to the higher dimensional case. I have placed integration after differentiation, because otherwise one has no technique available to evaluate integrals.

I personally think that the computations which arise naturally from Taylor's formula (computations of values of elementary functions, computation of e, π, $\log 2$, computations of definite integrals to a few decimals, traditionally slighted in calculus courses) are important. This was clear already many years ago, and is even clearer today in the light of the pocket computer proliferation. Designs of such computers rely precisely on effective means of computation by means of the Taylor polynomials. Learning how to estimate effectively the remainder term in Taylor's formula gives a very good feeling for the elementary functions, not obtainable otherwise.

The computation of integrals like

$$\int_0^1 e^{-x^2}\,dx \qquad \text{or} \qquad \int_0^{0.1} e^{-x^2}\,dx$$

which can easily be carried out numerically, without the use of a simple form for the indefinite integral, should also be emphasized. Again it gives a good feeling for an aspect of the integral not obtainable otherwise. Many texts slight these applications in favor of expanded treatment of applications of integration to various engineering situations, like fluid pressure on a dam, mainly by historical accident. I have nothing against fluid pressure, but one should keep in mind that too much time spent on some topics prevents adequate time being spent on others. For instance, Ron Infante tells me that numerical computations of integrals like

$$\int_0^1 \frac{\sin x}{x}\,dx,$$

which we carry out in Chapter XIII, occur frequently in the study of communication networks, in connection with square waves. Each instructor has to exercise some judgment as to what should be emphasized at the expense of something else.

The chapters on functions of several variables are included for classes which can proceed at a faster rate, and therefore have time for additional material during the first year. **Under ordinary circumstances, these chapters will not be covered during a first-year course. For instance, they are not covered during the first-year course at Yale.**

Induction. I think the first course in calculus is a good time to learn induction. However, an attempt to teach induction without having met natural examples first meets with very great psychological difficulties. Hence throughout the part on differentiation, I have not mentioned induction formally. Whenever a situation arises where induction may be used, I carry out stepwise procedures illustrating the inductive procedure. After enough repetitions of these, the student is then ready to see a pattern which can be summarized by the formal "induction," which just becomes a name given to a notion which has already been understood.

Review material. The present edition also emphasizes more review material. Deficient high school training is responsible for many of the difficulties experienced at the college level. These difficulties are not so much due to the problem of understanding calculus as to the inability to handle elementary algebra. A large group of students cannot automatically give the expansion for expressions like

$$(a+b)^2, \qquad (a-b)^2, \qquad \text{or} \qquad (a+b)(a-b).$$

The answers should be memorized like the multiplication table. To memorize by rote such basic formulas is not incompatible with learning general principles. It is complementary.

To avoid any misunderstandings, I wish to state explicitly that the poor preparation of so many high school students cannot be attributed to the "new math" versus the "old math." When I started teaching calculus as a graduate student in 1950, I found the quasi-totality of college freshmen badly prepared. Today, I find only a substantial number of them (it is hard to measure how many). On the other hand, a sizable group at the top has had the opportunity to learn some calculus, even as much as one year, which would have been inconceivable in former times. As bad as the situation is, it is nevertheless an improvement.

I wish to thank my colleagues at Yale and others in the past who have suggested improvements in the book: Edward Bierstone (University of Toronto), Folke Eriksson (University of Gothenburg), R. W. Gatterdam (University of Wisconsin, Parkside), and George Metakides (University of Rochester). I thank Ron Infante for assisting with the proofreading.

I am much indebted to Anthony Petrello for checking worked-out examples and answers in past editions. I am also much indebted to Allan Altman for a long list of corrections.

S. Lang

Contents