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T.Y. Lam

Exercises in Classical Ring Theory

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To Chee King

Juwen, Fumei, Juleen and Dee-Dee

Preface

This is a book I wished I had found when, many years ago, I first learned the subject of ring theory. All those years have flown by, but I still did not find *that* book. So finally I decided to write it myself.

All the books I have written so far were developed from my lectures; this one is no exception. After writing "*A First Course in Noncommutative Rings*" (Springer-Verlag GTM **131**, hereafter referred to as "*FC*"), I taught ring theory in Berkeley again in the fall of 1993, using *FC* as text. Since the main theory is already fully developed in *FC*, I asked my students to read the book at home, so that we could use part of the class time for doing the exercises from *FC*. The combination of lectures and problem sessions turned out to be a great success. By the end of the course, we covered a significant portion of *FC* and solved a good number of problems. There were 329 exercises in *FC* ; while teaching the course, I compiled 71 additional ones. The resulting four hundred exercises, with their full solutions, comprise this ring theory problem book.

There are many good reasons for a problem book to be written in ring theory, or for that matter in any subject of mathematics. First, the solutions to different exercises serve to illustrate the problem-solving process and show how general theorems in ring theory are applied in special situations. Second, the compilation of solutions to interesting and unusual exercises extends and completes the standard treatment of the subject in textbooks. Last, but not least, a problem book provides a natural place in which to record leisurely some of the folklore of the subject: the "tricks of the trade" in ring theory, which are well known to the experts in the field but may not be familiar to others, and for which there is usually no good reference. With all of the above objectives in mind, I offer this modest problem book for the use and enjoyment of students, teachers and researchers in ring theory and other closely allied fields.

This book is organized in the same way as *FC*, in eight chapters and twenty-five sections. It deals mainly with the "classical" parts of ring theory, starting with the Wedderburn-Artin theory of semisimple rings, Jacobson's theory of the radical, and the representation theory of groups and algebras, then continuing with prime and semiprime rings, primitive and semiprimitive rings, division rings, ordered rings, local and semilocal rings, and the theory of idempotents, and ending with perfect and semiperfect rings. For the reader's information, we should note that this book does not include problems in the vast areas of module theory (e.g., projectivity, injectivity, and flatness), category theory (e.g., equivalences and dualities), or rings of quotients (e.g., Ore rings and Goldie rings). A selection of exercises in these areas will appear later in the author's "*Second Course in Noncommutative Rings*".

While many problems in this book are chosen from *FC*, an effort has been made to render them as independent as possible from the latter. In particular, the statements of all problems are complete and self-contained and should be accessible to readers familiar with the subject at hand, either through *FC* or another ring theory text at the same level. But of course, solving ring theory problems requires a considerable tool kit of theorems and results. For such, I find it convenient to rely on the basic exposition in *FC*. (Results therein will be referred to in the form *FC*-(x.y).) For readers who may be using this book independently of *FC*, an additional challenge will be, indeed, to try to follow the proposed solutions and to figure out along the way *exactly what* are the theorems in *FC* needed to justify the different steps in a problem solution! Very possibly, meeting this challenge will be as rewarding an experience as solving the problem itself.

For the reader's convenience, each section in this book begins with a short introduction giving the general background and the theoretical basis for the problems that follow. All problems are solved in full, in most cases with a lot more details than can be found in the original sources (if they exist). A majority of the solutions are accompanied by a "*Comment*" section giving relevant bibliographical, historical or anecdotal information, pointing out relations to other exercises, or offering ideas on further improvements and generalizations. These "*Comment*" sections rounding out the solutions of the exercises are intended to be a particularly useful feature of this problem book.

The exercises in this book are of varying degrees of difficulty. Some are fairly routine and can be solved in a few lines. Others might require a good deal of thought and take up to a page for their solution. A handful of problems are chosen from research papers in the literature; the solutions of some of these might take a couple of pages. Problems of this latter kind are usually identified by giving an attribution to their sources. A majority of the other problems are from the folklore of the subject; with these, no attempt is made to trace the results to their origin. Thus, the lack of a reference for any particular problem only reflects my opinion that the problem is "in the public domain," and should in no case be construed as a claim to originality. On the other hand, the responsibility for any errors or flaws in the presentation of the solutions to any problems remains squarely my own. In the future, I would indeed very much like to receive from my readers communications concerning misprints, corrections, alternative solutions, etc., so that these can be taken into account in case later editions are possible.

Writing solutions to 400 ring-theoretic exercises was a daunting task, even though I had the advantage of choosing them in the first place. The arduous process of working out and checking these solutions could not have been completed without the help of others. Notes on many of the problems were distributed to my Berkeley class of fall 1993; I thank all students in this class for reading and checking my notes and making contributions toward the solutions. Dan Shapiro and Jean-Pierre Tignol have both given their time generously to this project, not only by checking some of my solutions but also by making many valuable suggestions for improvements. Their mathematical insights have greatly enhanced the quality of this work. Other colleagues have helped by providing examples and counterexamples, suggesting alternative solutions, pointing out references and answering my mathematical queries: among them, I should especially thank George Bergman, Rosa Camps, Keith Conrad, Warren Dicks, Kenneth Goodearl, Martin Isaacs, Irving Kaplansky, Hendrik Lenstra, André Leroy, Alun Morris and Barbara Osofsky. From start to finish, Tom von Foerster at Springer-Verlag has guided this project with a gentle hand; I remain deeply appreciative of his editorial acumen and thank him heartily for his kind cooperation.

As usual, members of my family deserve major credit for the timely completion of my work. The writing of this book called for no small sacrifices on the part of my wife Chee-King and our four children, Juwen, Fumei, Juleen, and Dee-Dee; it is thus only fitting that I dedicate this modest volume to them in appreciation for their patience, understanding and unswerving support.

Berkeley, California T.Y.L.
April, 1994

Notes to the Reader

The four hundred exercises in the eight chapters of this book are organized into twenty-five consecutively numbered sections. As we have explained in the Preface, many of these exercises are chosen from the author's *A First Course in Noncommutative Rings*, hereafter referred to as *FC*. A cross-reference such as *FC*-(12.7) refers to the result (12.7) in *FC*. Exercise 12.7 will refer to the exercise so labeled in §12 in this book. In referring to an exercise appearing (or to appear) in the same section, we shall sometimes drop the section number from the reference. Thus, when we refer to "Exercise 7" *within* §12, we shall mean Exercise 12.7.

The ring theory conventions used in this book are the same as those introduced in *FC*. Thus, a ring R means a ring with identity (unless otherwise specified). A subring of R means a subring containing the identity of R (unless otherwise specified). The word "ideal" always means a two-sided ideal; an adjective such as "noetherian" likewise means both right and left noetherian. A ring homomorphism from R to S is supposed to take the identity of R to that of S. Left and right R-modules are always assumed to be unital; homomorphisms between modules are (usually) written on the opposite side of the scalars. "Semisimple rings" are in the sense of Wedderburn, Noether and Artin: these are rings R that are semisimple as (left or right) modules over themselves. Rings with Jacobson radical zero are called Jacobson semisimple (or semiprimitive) rings.

Throughout the text, we use the standard notations of modern mathematics. For the reader's convenience, a partial list of the notations commonly used in basic algebra and ring theory is given on the following pages.

Some Frequently Used Notations

$\mathbb{Z}$	ring of integers
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$\mathbb{F}_q$	finite field with q elements
$\mathbb{M}_n(S)$	set of $n \times n$ matrices with entries from S
S_n	symmetric group on $\{1, 2, \dots, n\}$
A_n	alternating group on $\{1, 2, \dots, n\}$
$\subset, \subseteq$	used interchangeably for inclusion
$\subsetneqq$	strict inclusion
$\lvert A \rvert$, Card A	used interchangeably for the cardinality of the set A
$A \backslash B$	set-theoretic difference
$A \twoheadrightarrow B$	surjective mapping from A onto B
δ_{ij}	Kronecker deltas
E_{ij}	matrix units
tr	trace (of a matrix or a field element)
det	determinant of a matrix
$\langle x \rangle$	cyclic group generated by x
$Z(G)$	center of the group (or the ring) G
$C_G(A)$	centralizer of A in G
$H \lhd G$	H is a normal subgroup of G
$[G : H]$	index of subgroup H in a group G
$[K : F]$	field extension degree
$[K : D]_\ell$, $[K : D]_r$	left, right dimensions of $K \supseteq D$ as D-vector space
K^G	G-fixed points on K
$\text{Gal}(K/F)$	Galois group of the field extension K/F
M_R, ${}_RN$	right R-module M, left R-module N
$M \oplus N$	direct sum of M and N
$M \otimes_R N$	tensor product of M_R and ${}_RN$
$\text{Hom}_R(M, N)$	group of R-homomorphisms from M to N
$\text{End}_R(M)$	ring of R-endomorphisms of M
$\text{soc}(M)$	socle of M
$\text{length}(M)$, $\ell(M)$	(composition) length of M
nM (or M^n)	$M \oplus \cdots \oplus M$ (n times)
$\prod_i R_i$	direct product of the rings $\{R_i\}$
char R	characteristic of the ring R
R^{op}	opposite ring of R
$U(R)$, R^*	group of units of the ring R
$U(D)$, D^*, $\dot{D}$	multiplicative group of the division ring D
$GL_n(R)$	group of invertible $n \times n$ matrices over R

$GL(V)$	group of linear automorphisms of a vector space V
rad R	Jacobson radical of R
$\mathrm{Nil}^*(R)$	upper nilradical of R
$\mathrm{Nil}_*(R)$	lower nilradical (or prime radical) of R
Nil (R)	ideal of nilpotent elements in a commutative ring R
$\mathrm{soc}(_RR)$, $\mathrm{soc}(R_R)$	socle of R as left, right R-module
$\mathrm{ann}_\ell(S)$, $\mathrm{ann}_r(S)$	left, right annihilators of the set S
kG, $k[G]$	(semi)group ring of the (semi)group G over the ring k
$k[x_i : i \in I]$	polynomial ring over k with (commuting) variables $\{x_i : i \in I\}$
$k\langle x_i : i \in I\rangle$	free ring over k generated by $\{x_i : i \in I\}$
$k[x; \sigma]$	skew polynomial ring with respect to an endomorphism σ on k
$k[x; \delta]$	differential polynomial ring with respect to a derivation δ on k
$[G, G]$	commutator subgroup of the group G
$[R, R]$	additive subgroup of the ring R generated by all $[a, b] = ab - ba$
f.g.	finitely generated
ACC	ascending chain condition
DCC	descending chain condition
LHS	left-hand side
RHS	right-hand side

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