Graduate Texts in Mathematics **131**

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T. Y. Lam

A First Course in Noncommutative Rings

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To Juwen, Fumei, Juleen and Dee-Dee

who form a most delightful ring

Preface

One of my favorite graduate courses at Berkeley is Math 251, a one-semester course in ring theory offered to second-year level graduate students. I taught this course in the Fall of 1983, and more recently in the Spring of 1990, both times focusing on the theory of noncommutative rings. This book is an outgrowth of my lectures in these two courses, and is intended for use by instructors and graduate students in a similar one-semester course in basic ring theory.

Ring theory is a subject of central importance in algebra. Historically, some of the major discoveries in ring theory have helped shape the course of development of modern abstract algebra. Today, ring theory is a fertile meeting ground for group theory (group rings), representation theory (modules), functional analysis (operator algebras), Lie theory (enveloping algebras), algebraic geometry (finitely generated algebras, differential operators, invariant theory), arithmetic (orders, Brauer groups), universal algebra (varieties of rings), and homological algebra (cohomology of rings, projective modules, Grothendieck and higher K-groups). In view of these basic connections between ring theory and other branches of mathematics, it is perhaps no exaggeration to say that a course in ring theory is an indispensable part of the education for any fledgling algebraist.

The purpose of my lectures was to give a general introduction to the theory of rings, building on what the students have learned from a standard first-year graduate course in abstract algebra. We assume that, from such a course, the students would have been exposed to tensor products, chain conditions, some module theory, and a certain amount of commutative algebra. Starting with these prerequisites, I designed a course dealing almost exclusively with the theory of noncommutative rings. In accordance with the historical development of the subject, the course begins with the Wedderburn–Artin theory of semisimple rings, then goes on to Jacobson's general theory of the radical for rings possibly not satisfying any chain conditions. After an excursion into representation theory in the style of Emmy Noether, the course continues with the study of prime and semiprime rings, primitive and semiprimitive rings, division rings, ordered rings, local and semilocal rings, and finally, perfect and semiperfect rings. This material, which was as much as I managed to cover in a one-semester course, appears here in a somewhat expanded form as the eight chapters of this book.

Of course, the topics described above correspond only to part of the foundations of ring theory. After my course in Fall, 1983, a self-selected group of students from this course went on to take with me a second course (Math 274, Topics in Algebra), in which I taught some further basic topics in the subject. The notes for this second course, at present only partly written, will hopefully also appear in the future, as a sequel to the present work. This intended second volume will cover, among other things, the theory of modules, rings of quotients and Goldie's Theorem, noetherian rings, rings with polynomial identities, Brauer groups and the structure theory of finite-dimensional central simple algebras. The reasons for publishing the present volume first are two-fold: first it will give me the opportunity to class-test the second volume some more before it goes to press, and secondly, since the present volume is entirely self-contained and technically independent of what comes after, I believe it is of sufficient interest and merit to stand on its own.

Every author of a textbook in mathematics is faced with the inevitable challenge to do things differently from other authors who have written earlier on the same subject. But no doubt the number of available proofs for any given theorem is finite, and by definition the best approach to any specific body of mathematical knowledge is unique. Thus, no matter how hard an author strives to appear original, it is difficult for him to avoid a certain degree of "plagiarism" in the writing of a text. In the present case I am all the more painfully aware of this since the path to basic ring theory is so well-trodden, and so many good books have been written on the subject. If, of necessity, I have to borrow so heavily from these earlier books, what are the new features of this one to justify its existence?

In answer to this, I might offer the following comments. Although a good number of books have been written on ring theory, many of them are monographs devoted to specialized topics (e.g., group rings, division rings, noetherian rings, von Neumann regular rings, or module theory, PI-theory, radical theory, localization theory). A few others offer general surveys of the subject, and are encyclopedic in nature. If an instructor tries to look for an introductory graduate text for a one-semester (or two-semester) course in ring theory, the choices are still surprisingly few. It is hoped, therefore, that the present book (and its sequel) will add to this choice. By aiming the level of writing at the novice rather than the connoisseur, we have sought to produce a text which is suitable not only for use in a graduate course, but also for self-study in the subject by interested graduate students.

Since this book is a by-product of my lectures, it certainly reflects much more on my teaching style and my personal taste in ring theory than on ring theory itself. In a graduate course one has only a limited number of lectures at one's disposal, so there is the need to "get to the point" as quickly as possible in the presentation of any material. This perhaps explains the often business-like style in the resulting lecture notes appearing here. Nevertheless, we are fully cognizant of the importance of motivation and examples, and we have tried hard to ensure that they don't play second fiddle to theorems and proofs. As far as the choice of the material is concerned, we have perhaps given more than the usual emphasis to a few of the famous open problems in ring theory, for instance, the Köthe Conjecture for rings with zero upper nilradical (§10), the semiprimitivity problem and the zero-divisor problem for group rings (§6), etc. The fact that these natural and very easily stated problems have remained unsolved for so long seemed to have captured the students' imagination. A few other possibly "unusual" topics are included in the text: for instance, noncommutative ordered rings are treated in §17, and a detailed exposition of the Mal'cev–Neumann construction of general Laurent series rings is given in §14. Such material is not easily available in standard textbooks on ring theory, so we hope its inclusion here will be a useful addition to the literature.

There are altogether twenty five sections in this book, which are consecutively numbered independently of the chapters. Results in Section x will be labeled in the form (x.y). Each section is equipped with a collection of exercises at the end. In almost all cases, the exercises are perfectly "doable" problems which build on the text material in the same section. Some exercises are accompanied by copious hints; however, the more self-reliant readers should not feel obliged to use these.

As I have mentioned before, in writing up these lecture notes I have consulted extensively the existing books on ring theory, and drawn material from them freely. Thus I owe a great literary debt to many earlier authors in the field. My graduate classes in Fall 1983 and Spring 1990 at Berkeley were attended by many excellent students; their enthusiasm for ring theory made the class a joy to teach, and their vigilance has helped save me from many slips. I take this opportunity to express my appreciation for the role they played in making these notes possible. A number of friends and colleagues have given their time generously to help me with the manuscript. It is my great pleasure to thank especially Detlev Hoffmann, André Leroy, Ka-Hin Leung, Mike May, Dan Shapiro, Tara Smith and Jean-Pierre Tignol for their valuable comments, suggestions, and corrections. Of course, the responsibility for any flaws or inaccuracies in the exposition remains my own. As mathematics editor at Springer-Verlag, Ulrike Schmickler-Hirzebruch has been most understanding of an author's plight, and deserves a word of special thanks for bringing this long overdue project to fruition. Keyboarder Kate MacDougall did an excellent job in transforming my handwritten manuscript into LaTex, and the Production Department's efficient handling of the entire project has been exemplary.

Last, first, and always, I owe the greatest debt to members of my family. My wife Chee-King graciously endured yet another book project, and our four children bring cheers and joy into my life. Whatever inner strength I can muster in my various endeavors is in large measure a result of their love, devotion, and unstinting support.

T.Y.L.

Berkeley, California
November, 1990

Contents

Notes to the Reader

As we have explained in the Preface, the twenty five sections in this book are numbered independently of the eight chapters. A cross-reference such as (12.7) refers to the result so labeled in §12. On the other hand, Exercise 12.7 will refer to Exercise 7 appearing at the end of §12. In referring to an exercise appearing (or to appear) in the same section, we shall sometimes drop the section number from the reference. Thus, when we refer to "Exercise 7" anywhere *within* §12, we shall mean Exercise 12.7.

Since this is an exposition and not a treatise, the writing is by no means encyclopedic. In particular, in most places, no systematic attempt is made to give attributions, or to trace the results discussed to their original sources. References to a book or a paper are given only sporadically where they seem more essential to the material under consideration. A reference in brackets such as Amitsur [56] (or [Amitsur: 56]) shall refer to the 1956 paper of Amitsur listed in the reference section at the end of the book.

Occasionally, references will be made to the intended sequel of this book, which will be briefly called *Second Course.* Such references will always be peripheral in nature; their only purpose is to point to material which lies ahead. In particular, no result in this book will depend logically on any result to appear later in *Second Course.*

Throughout the text, we use the standard notations of modern mathematics. For the reader's convenience, a partial list of the notations commonly used in basic algebra and ring theory is given on the following pages.

Some Frequently Used Notations

$\mathbb{Z}$	ring of integers
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$\mathbb{F}_q$	finite field with q elements
$\mathbb{M}_n(S)$	set of $n \times n$ matrices with entries from S
$\subset$, $\subseteq$	used interchangeably for inclusion
$\subsetneq$	strict inclusion
$\lvert A\rvert$, $Card\ A$	used interchangeably for the cardinality of the set A
$A \setminus B$	set-theoretic difference
$A \twoheadrightarrow B$	surjective mapping from A onto B
δ_{ij}	Kronecker deltas
E_{ij}	matrix units
tr	trace (of a matrix or a field element)
$\langle x \rangle$	cyclic group generated by x
$Z(G)$	center of the group (or the ring) G
$C_G(A)$	centralizer of A in G
$[G : H]$	index of subgroup H in a group G
$[K : F]$	field extension degree
$[K : D]_\ell$, $[K : D]_r$	left, right dimensions of $K \supseteq D$ as D-vector space
K^G	G-fixed points on K
M_R, ${}_RN$	right R-module M, left R-module N
$M \otimes_R N$	tensor product of M_R and ${}_RN$
$Hom_R(M, N)$	group of R-homomorphisms from M to N
$End_R(M)$	ring of R-endomorphisms of M
nM (or M^n)	$M \oplus \cdots \oplus M$ (n times)
$\prod_i R_i$	direct product of the rings $\{R_i\}$
$char\ R$	characteristic of a ring R
$U(R)$, R^*	group of units of the ring R
$U(D)$, D^*, $\dot{D}$	multiplicative group of the division ring D
$GL_n(R)$	group of invertible $n \times n$ matrices over R
$GL(V)$	group of linear automorphisms of a vector space V
$rad\ R$	Jacobson radical of R
$Nil^*(R)$	upper nilradical of R
$Nil_*(R)$	lower nilradical (or prime radical) of R
$Nil\ R$	ideal of nilpotent elements in a commutative ring R
$ann_\ell(S)$, $ann_r(S)$	left, right annihilators of the set S

kG, $k[G]$	(semi)group ring of the (semi)group G over the ring k
$k[x_i : i \in I]$	polynomial ring over k with (commuting) variables $\{x_i : i \in I\}$
$k\langle x_i : i \in I\rangle$	free ring over k generated by $\{x_i : i \in I\}$
ACC	ascending chain condition
DCC	descending chain condition
LHS	left-hand side
RHS	right-hand side