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# Norm Inequalities for Derivatives and Differences

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## Preface

Edmund Landau's 1913 paper "Einige Ungleichungen für zweimal differenzierbare Funktionen", based on earlier work of Hardy and Littlewood, initiated a vast and fruitful research activity involving the study of the relationship between the norms of (i) a function and its derivatives and (ii) a sequence and its differences. These notes are an attempt to give a connected account of this effort. Detailed elementary proofs of basic inequalities are given. These are accessible to anyone with a background of advanced calculus and a rudimentary knowledge of the $L^p$ and $l^p$ spaces, yet the reader will be brought to the frontier of knowledge regarding several aspects of these problems. Many open questions are raised.

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