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Algebra VI

Combinatorial and Asymptotic Methods of Algebra. Non-Associative Structures

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I. Combinatorial and Asymptotic Methods in Algebra

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