Measure and Integration

Heinz König

Measure and Integration

An Advanced Course in Basic Procedures and Applications

Springer

Heinz König
Universität des Saarlandes
Fakultät für Mathematik und Informatik
D-66123 Saarbrücken
Germany
hkoenig@math.uni-sb.de

The cover picture shows a section of the skeleton of a marine diatomee arachnoidiscus. (SEM micrograph reproduced by courtesy of Manfred P. Kage).
This is a beautiful example of structure in nature, which reoccurs in man-made buildings and is a model for structure in mathematics.

ISBN 978-3-540-61858-4 e-ISBN 978-3-540-89502-2

DOI 10.1007/978-3-540-89502-2

Library of Congress Cataloging-in-Publication Data available.

Die Deutsche Bibliothek – CIP-Einheitsaufnahme
König, Heinz: Measure and integration: an advanced course in basic procedures and applications/Heinz König. – Berlin; Heidelberg; New York; Barcelona; Budapest; Hong Kong; London; Milan; Paris; Santa Clara; Singapore; Tokyo: Springer, 1997

Mathematics Subject Classification (2000): 28-02

Corrected, 2nd printing 2009

Cover design: WMX Design GmbH, Heidelberg

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

Meiner Frau Karin
in Liebe und Dankbarkeit gewidmet

Preface

Ich schaffe, was ihr wollt, und schaffe mehr;
Zwar ist es leicht, doch ist das Leichte schwer.
Es liegt schon da, doch um es zu erlangen,
Das ist die Kunst! Wer weiss es anzufangen?
GOETHE, Faust II

The present text centers around a fundamental task of measure and integration theory, which has not found an adequate solution so far. It is the task to produce, with unified and universal means, true contents and above all measures from more primitive data, in order to extend elementary contents and to represent so-called elementary integrals. The traditional main tools are the Carathéodory extension theorem and the Daniell-Stone representation theorem. These theorems are much too restrictive in order to fulfil the needs.

Around 1970 a new development started in the work of Topsøe and others. It was based on the notion of regularity, which for a set function means to determine its values from a particular set system by approximation from above or below. In traditional measure theory this notion is linked to topology.

The present text wants to be a systematic treatment of the context in the new spirit. It is based to some extent on personal work of the author. The main results are equivalence theorems for the existence and uniqueness of extensions and representations, which are not more complicated than the traditional ones but much more powerful. With these results the text clarifies and unifies the entire context. The main instruments are certain new envelope formations which resemble the traditional Carathéodory outer measure.

The systematic theory has numerous applications. The most important application is the full extension of the classical Riesz representation theorem in terms of Radon measures, from locally compact to arbitrary Hausdorff topological spaces. As another application we note an extension and at the same time simplification of the Choquet capacitability theorem, which shows that the new formations can be useful for so-called non-additive set functions as well. Some of the applications are treated without pronounced technical sophistication. We rather want to demonstrate that certain basic ideas and results are natural outflows from the new theory.

The central parts of the text are chapters II and V. Their main substance as well as their history and motivation are outlined in the introduction below. It is an elaboration of a lecture which the author delivered at several places, in the present form for the first time at the symposium in honour of Adriaan C.Zaanen in Leiden in September 1993.

Chapters I and IV are filled with preparations. We need certain standard material in unconventional versions which have to be developed. We also need several new notations.

The application to the Riesz representation theorem is in chapter V section 16. The other applications are in chapters III, VI and VII. We emphasize that chapter VII develops an abstract product formation which comprises the Radon product measure of Radon measures. The final chapter VIII is an appendix which is independent of the central chapters II and V. It wants to demonstrate that the unconventional notions of content and measure introduced in chapter I can be useful in other areas of measure theory as well.

All this says that the central themes of the present text are the fundamentals of measure and integration theory. The author hopes that its readers will find it less technical than it looks at first sight. He thinks that the text can be read with appreciation by anyone who has struggled through the traditional abstract and topological theories. However, it is different from a textbook in the usual sense. The presentation is ab ovo, though more like in a book of research. The author hopes that the text will be used in future courses. An ideal prerequisite would be the recent small book of Stroock [1994], because on the one hand it provides the concrete material which should precede this one, and on the other hand it does not take the reader onto the traditional paths of abstract measure and integration theory which the present work wants to restructure.

The author wants to express his warmest thanks to Gustave Choquet, Jean-Paul Pier, Reinhold Remmert, Klaus D.Schmidt, Maurice Sion, and Flemming Topsøe for insightful comments, encouragement, and good advice. Likewise he thanks Robert Berger and Gerd Wittstock for constant help with the resistful machine into which he typed the final version of the text. He extends his thanks to the former and present directors of the Mathematical Research Institute Oberwolfach, Martin Barner and Matthias Kreck, for several periods of quiet work in the unique atmosphere of the Institute.

August 1996 Heinz König

Contents

Introduction

The textbooks on measure and integration theory can often be subdivided into two parts of almost equal size: One part describes what can be done when one is in possession of measures of one or another type. The other part describes how to obtain these measures from more primitive data, which as a rule are elementary contents or elementary integrals. The former part is based on some famous theories. But the latter part is in less favourable state, because its main theorems do not fit the actual needs in certain central points. We shall explain this statement, and then describe how the situation can be repaired. To do this we sketch the main ideas and results of our chapters II and V, which form the central parts of the present text.

Construction of Measures from Elementary Contents

The classical theorem on the existence of measure extensions reads as follows. Our technical terms are either familiar or obvious.

THEOREM. *Let* $\varphi : \mathfrak{S} \to [0, \infty]$ *be a content on a ring* $\mathfrak{S}$ *of subsets in a nonvoid set* X. *Then* φ *can be extended to a measure* $\alpha : \mathfrak{A} \to [0, \infty]$ *on a* σ *algebra* $\mathfrak{A}$ *iff* φ *is upward* σ *continuous.*

There are few situations where this theorem can be applied without complications. The reason is that the natural set systems which carry elementary contents are almost never rings, but at most lattices. This is in particular true for the basic set systems in topological spaces. Even to construct the Lebesgue measure via rings forces us to work with the unnatural half-open intervals, which might be adequate in order to produce sophisticated counterexamples, but not for the foundations of one of the most basic theories in analysis.

Like the theorem itself, also its usual proof due to Carathéodory [1914] does not fit the actual needs as it stands. Let us recall that it is based on two formations. On the one hand one defines for a set function $\varphi : \mathfrak{S} \to [0, \infty]$ on a set system $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$ the so-called **outer measure** $\varphi^\circ : \mathfrak{P}(X) \to [0, \infty]$ to be

$$\varphi^\circ(A) = \inf\Big\{\sum_{l=1}^{\infty}\varphi(S_l) : (S_l)_l \text{ in } \mathfrak{S} \text{ with } A \subset \bigcup_{l=1}^{\infty} S_l\Big\}.$$

On the other hand one defines for a set function $\phi : \mathfrak{P}(X) \to [0,\infty]$ with $\phi(\varnothing) = 0$ the so-called **Carathéodory class**

$$\mathfrak{C}(\phi) := \{A \subset X : \phi(S) = \phi(S \cap A) + \phi(S \cap A') \ \forall S \subset X\} \subset \mathfrak{P}(X).$$

Then for the nontrivial direction of the theorem one verifies that $\varphi^\circ|\mathfrak{C}(\varphi^\circ)$ is a measure on a σ algebra and an extension of φ.

We shall see that the formation $\mathfrak{C}(\cdot)$ is so felicitous that it will survive the upheaval to come, at least within the present step of abstraction. In contrast, we shall see that the specific form of the outer measure must be blamed for the deficiencies around the extension theorem which will now be described in more detail.

1) The outer measure is a beautiful tool in the frame of rings, but it ceases to work beyond this frame. It does not even allow to extend the theorem to the particular lattices $\mathfrak{S}$ which fulfil $B \setminus A \in \mathfrak{S}^\sigma$ for all $A \subset B$ in $\mathfrak{S}$, where the assertion will be seen to persist. The class of these lattices is much more realistic than the class of rings. For example, it includes the lattices of the closed subsets and of the compact subsets of a metric space.

2) The outer measure is an outer regular formation: The definition shows that

$$\varphi^\circ(A) = \inf\{\varphi^\circ(S) : S \in \mathfrak{S}^\sigma \text{ with } S \supset A\} \quad \text{for all } A \subset X,$$

that is φ° is **outer regular $\mathfrak{S}^\sigma$**. Now present-day analysis requires inner regular formations perhaps even more than outer regular ones. However, the definition of the outer measure is such that no inner regular counterpart is visible.

The need for inner regular formations comes from the predominant role of compactness in topological measure theory. It became clear that the most important class of measures on an arbitrary Hausdorff topological space X are the **Radon measures**, defined to be the Borel measures $\alpha : \mathrm{Bor}(X) \to [0,\infty]$ which are finite on the system $\mathrm{Comp}(X)$ of the compact subsets of X and inner regular $\mathrm{Comp}(X)$. It is then an immediate problem to characterize those set functions $\varphi : \mathrm{Comp}(X) \to [0,\infty[$ which can be extended to (of course unique) Radon measures, the so-called **Radon premeasures**. We see that the classical extension theorem does not help in this problem for at least two reasons.

3) The outer measure is a formation of *sequential* type. But present-day analysis also requires non-sequential formations, once more for topological reasons. However, the definition of the outer measure is such that no non-sequential counterpart is visible.

4) It is a sad fact that the methods employed for contents and measures have not much in common with those for so-called *non-additive* set functions like capacities. Now the outer measure has a certain built-in *additive* character. One can be suspicious that this fact is responsible for the imperfections which we speak about.

There were of course attempts to improve the situation. The main results of Pettis [1951] were complicated and hard to use because, as it seems now, regularity had not yet attained its true position. Srinivasan [1955] was restricted to the extension from rings, but was able to develop a symmetric outer/inner extension procedure and anticipated the later expressions in this frame. Around 1970 deliberate efforts started in order to develop improved extension methods in terms of lattices, outer and inner regularity, and sequential and non-sequential procedures. A decisive prelude was the characterization of the Radon premeasures due to Kisyński [1968]. The main achievements came from Topsøe [1970ab], albeit restricted to the inner situation, from Kelley-Srinivasan [1971] and Kelley-Nayak-Srinivasan [1973], Ridder [1971][1973], and later from Sapounakis-Sion [1983][1987] and others. But the new methods were less simple and coherent than the traditional ones and therefore did not find access to the textbooks. The reason was that there were no universal substitutes for the outer measure. It is a surprise that one did not resume the expressions of Srinivasan [1955] (as a result the author himself did not look at that paper earlier than while he wrote the present text). Also there was no adequate symmetric treatment of the outer and inner cases. The basic symmetric formations were the crude outer and inner envelopes $\varphi^\star, \varphi_\star : \mathfrak{P}(X) \rightarrow [0, \infty]$, defined for an isotone set function $\varphi : \mathfrak{S} \rightarrow [0, \infty]$ with $\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$ to be

$$\begin{aligned} \varphi^\star(A) &= \inf\{\varphi(S) : S \in \mathfrak{S} \text{ with } S \supset A\}, \\ \varphi_\star(A) &= \sup\{\varphi(S) : S \in \mathfrak{S} \text{ with } S \subset A\}, \end{aligned}$$

which are adequate for contents but not for measures (otherwise the outer measure would not have come into existence).

At this point we postpone further historical comments and turn to the vita of the present author on which the plan for this text is based. In an analysis course [1969/70] I wanted to construct the Lebesgue measure without use of half-open intervals. I observed that the old proof extends without further efforts from rings to the particular lattices described in 1), provided that instead of the outer measure one uses the formation $\varphi^\sigma : \mathfrak{P}(X) \rightarrow [0, \infty]$, defined for an isotone set function $\varphi : \mathfrak{S} \rightarrow [0, \infty]$ to be

$$\varphi^\sigma(A) = \inf\Big\{\lim_{l\to\infty} \varphi(S_l) : (S_l)_l \text{ in } \mathfrak{S} \text{ with } S_l \uparrow \text{ some subset } \supset A\Big\}.$$

The formations φ° and φ^σ are of course close relatives, and are in fact identical for contents on rings (as in elementary analysis infinite series are equivalent to infinite sequences), but need not be identical beyond. We see that φ^σ continues to work where φ° does not.

At that time I was content with this. But fifteen years later I returned to the context and observed that besides 1) the new formation also removes the deficiencies described in 2) and 3). In fact, the formation φ^σ has an obvious inner regular counterpart $\varphi_\sigma : \mathfrak{P}(X) \rightarrow [0, \infty]$, defined via decreasing sequences in $\mathfrak{S}$. Furthermore φ^σ and φ_σ have obvious non-sequential counterparts $\varphi^\tau, \varphi_\tau : \mathfrak{P}(X) \to [0,\infty]$, defined via upward/downward directed set systems instead of sequences in $\mathfrak{S}$. Then another five years later I observed that the new formations permit to improve certain concepts and results related to capacities, and thus contribute to 4) as well.

After this it is no surprise that the envelope formations $\varphi^\star \geqq \varphi^\sigma \geqq \varphi^\tau$ and $\varphi_\star \leqq \varphi_\sigma \leqq \varphi_\tau$ permit to develop comprehensive extension theories which fulfil the requirements described above. The theories are of uniform structure in $\bullet = \star\sigma\tau$, and the outer and inner developments are parallel in all essentials. For historical reasons the outer version looks more familiar, but the inner version is perhaps more important. The Carathéodory class $\mathfrak{C}(\cdot)$ is a basic notion in all cases.

There remains one more step. I observed that the outer and inner theories are not only parallel, with their typical little peculiarities, but are in fact identical. However, this presupposes a drastic step of extension and abstraction: One has to admit lattices which avoid the empty set like the entire space, and isotone set functions with values in $\mathbb{R}$ or $\overline{\mathbb{R}}$ instead of $[0,\infty[$ or $[0,\infty]$ (not to be confused with the familiar signed measures which of course need not be isotone). The previous envelope formations retain their basic structure, but the Carathéodory class $\mathfrak{C}(\cdot)$ requires an essential reformulation. I consider this extension to be quite essential for theoretical reasons, but it is too technical for an introduction. Thus we return to the previous step. We choose the inner situation for a short description of the basic concepts and results.

Let $\varphi : \mathfrak{S} \to [0,\infty[$ be an isotone set function on a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ and $\varphi(\varnothing) = 0$. The basic idea is to concentrate on a particular class of extensions of φ. For each choice of $\bullet = \star\sigma\tau$ we define an **inner $\bullet$ extension** of φ to be an extension of φ which is a content $\alpha : \mathfrak{A} \to [0,\infty]$ on a ring $\mathfrak{A}$, with the properties that $\mathfrak{A}$ also contains $\mathfrak{S}_\bullet$ (:=the system of the respective intersections), and that

α is inner regular $\mathfrak{S}_\bullet$,
$\alpha|\mathfrak{S}_\bullet$ is downward $\bullet$ continuous (this is void when $\bullet = \star$).

Thus we impose a characteristic combination of inner regularity and downward continuity. We define φ to be an **inner $\bullet$ premeasure** iff it admits inner $\bullet$ extensions. Our aim is to characterize those φ which are inner $\bullet$ premeasures, and then to describe all inner $\bullet$ extensions of φ. We shall obtain a natural and beautiful solution.

The solution will be in terms of the inner envelopes $\varphi_\bullet : \mathfrak{P}(X) \to [0,\infty]$. First note that $\varphi_\star|\mathfrak{S} = \varphi$, while for $\bullet = \sigma\tau$ we have $\varphi_\bullet|\mathfrak{S} = \varphi$ iff φ is downward $\bullet$ continuous. This is of course a necessary condition in order that φ be an inner $\bullet$ premeasure. Likewise $\varphi_\bullet(\varnothing) = 0$ iff φ is (of course downward) $\bullet$ continuous at $\varnothing$. This weaker condition is much easier and sometimes even trivial, for example when $\varphi : \mathrm{Comp}(X) \to [0,\infty[$ on a Hausdorff topological space X. Also $\varphi_\bullet(\varnothing) = 0$ ensures that the traditional $\mathfrak{C}(\varphi_\bullet)$ is defined. We turn to the main results.

PROPOSITION. *If φ has inner $\bullet$ extensions then all these $\alpha : \mathfrak{A} \to [0,\infty]$ are restrictions of $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$.*

THEOREM. *Assume that φ is supermodular. Then the following are equivalent.*

1) *φ has inner $\bullet$ extensions, that is φ is an inner $\bullet$ premeasure.*

2) $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ *is* (defined and) *an inner $\bullet$ extension of φ. Furthermore*

$$\begin{aligned} \textit{if } \bullet = \star \quad &: \quad \varphi_\bullet|\mathfrak{C}(\varphi_\bullet) \textit{ is a content on the algebra } \mathfrak{C}(\varphi_\bullet), \\ \textit{if } \bullet = \sigma\tau \quad &: \quad \varphi_\bullet|\mathfrak{C}(\varphi_\bullet) \textit{ is a measure on the } \sigma \textit{ algebra } \mathfrak{C}(\varphi_\bullet). \end{aligned}$$

3) $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ *is* (defined and) *an extension of φ in the crude sense, that is $\varphi_\bullet|\mathfrak{S} = \varphi$ and $\mathfrak{S} \subset \mathfrak{C}(\varphi_\bullet)$.*

4) *$\varphi(B) = \varphi(A) + \varphi_\bullet(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*

5) *$\varphi_\bullet|\mathfrak{S} = \varphi$; and $\varphi(B) \leqq \varphi(A) + \varphi_\bullet(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.*

5') *$\varphi_\bullet(\varnothing) = 0$; and $\varphi(B) \leqq \varphi(A) + \varphi_\bullet^B(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$.* Here $\varphi_\bullet^B := \big(\varphi|\{S \in \mathfrak{S} : S \subset B\}\big)_\bullet$ for $B \in \mathfrak{S}$.

We define φ to be **inner $\bullet$ tight** iff it fulfils the second partial condition in 5').

It follows that an inner $\bullet$ premeasure φ has a unique maximal inner $\bullet$ extension, which is $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$. The above theorem and its outer counterpart are our substitutes for the classical extension theorem. It is obvious that the present characterizations and explicit representations stand and fall with the new envelopes.

Construction of Measures from Elementary Integrals

This time we start with the traditional Daniell-Stone representation theorem. It is the counterpart and also an extension of the classical measure extension theorem.

THEOREM. *Let $I : H \to \mathbb{R}$ be a positive* (:=isotone) *linear functional on a Stonean lattice subspace $H \subset \mathbb{R}^X$ of real-valued functions on a nonvoid set X. Then the following are equivalent.*

i) *There exists a measure $\alpha : \mathfrak{A} \to [0,\infty]$ on a σ algebra $\mathfrak{A}$ which represents I, that is all $f \in H$ are integrable α with $I(f) = \int f d\alpha$.*

ii) *I is σ continuous at 0, that is for each sequence $(f_l)_l$ in H with pointwise $f_l \downarrow 0$ one has $I(f_l) \downarrow 0$.*

More famous than this is perhaps the traditional Riesz representation theorem from topological measure theory.

THEOREM. *Let X be a locally compact Hausdorff topological space, and*

$$\mathrm{CK}(X,\mathbb{R}) := \{f \in \mathrm{C}(X,\mathbb{R}) : f = 0 \textit{ outside of some } K \in \mathrm{Comp}(X)\}.$$

Then there is a one-to-one correspondence between the positive linear functionals $I : \mathrm{CK}(X,\mathbb{R}) \to \mathbb{R}$ and the Radon measures $\alpha : \mathrm{Bor}(X) \to [0,\infty]$.

The correspondence is

$$I(f) = \int f d\alpha \quad \textit{for all } f \in \mathrm{CK}(X, \mathbb{R}).$$

The drawbacks of the traditional Daniell-Stone theorem are like those of the classical measure extension theorem. Thus it is of no visible use for the proof of the traditional Riesz theorem. But this latter theorem does not fulfil the needs either, because in present-day analysis one is often forced to exceed the frame of local compactness. Then $\mathrm{CK}(X, \mathbb{R})$ becomes too small, so that the theorem breaks down and has to be filled with new substance. On the measure side one wants to adhere to the Radon measures. As to the functional side, one observes that on each Hausdorff topological space X there is a wealth of *semicontinuous* real-valued functions which vanish outside of compact subsets, for example the multiples of the characteristic functions χ_K of the $K \in \mathrm{Comp}(X)$. But this leads to function classes which are *lattice cones* and as a rule not lattice subspaces. Thus it seems natural to search for an extended Riesz theorem on appropriate lattice cones of upper semicontinuous functions on X with values in $[0, \infty[$.

With this in mind we return to the Daniell-Stone theorem in the abstract theory. We want to develop the context in the spirit and scope of the previous part on measure extensions. The above look at the Riesz theorem confirms our intuitive impression that the former transition from rings to lattices should reappear as a transition from lattice subspaces to lattice cones. In fact, we shall see that the final Riesz theorem will become a direct specialization of the final Daniell-Stone theorem.

We fix a lattice cone $E \subset [0, \infty[^X$ of $[0, \infty[$-valued functions on a nonvoid set X. E is called **primitive** iff $v - u \in E$ for all $u \leqq v$ in E; equivalent is $E = H^+ := \{f \in H : f \geqq 0\}$ for some (unique) lattice subspace $H \subset \mathbb{R}^X$. It is of utmost importance that E need not be primitive. We assume E to be *Stonean*, defined to mean that $f \in E \Rightarrow f \wedge t, (f - t)^+ \in E$ for all real $t > 0$. In view of $f = f \wedge t + (f - t)^+$ this is the familiar notion when E is primitive. For E we define at once the set system

$$\mathfrak{T}(E) := \{[f \geqq t] : f \in E \text{ and } t > 0\} = \{[f \geqq 1] : f \in E\},$$

which is a lattice with $\varnothing \in \mathfrak{T}(E)$.

Next we fix an **elementary integral** on E, defined to be an isotone positive-linear functional $I : E \to [0, \infty[$. We are interested in integral representations of I. We want to define a **representation** of I to be a content $\alpha : \mathfrak{A} \to [0, \infty]$ on a ring $\mathfrak{A}$ such that

$$\text{for all } f \in E : f \text{ is measurable } \mathfrak{A} \text{ and } I(f) = \int f d\alpha.$$

This has to be made precise, except in the special case that $\mathfrak{A}$ is a σ algebra and α is a measure. We do this in that we require

$$\text{for all } f \in E : [f \geqq t] \in \mathfrak{A}\ \forall t > 0 \text{ and } I(f) = \int_{0\leftarrow}^{\rightarrow\infty} \alpha([f \geqq t])dt.$$

The first part of the condition means that $\mathfrak{T}(E) \subset \mathfrak{A}$. Therefore α produces the restriction $\alpha|\mathfrak{T}(E)$. The set function $\alpha|\mathfrak{T}(E)$ is of obvious importance, because it suffices to reproduce I by the second part of the condition.

It is a Hahn-Banach consequence that I admits representations iff it has the truncation properties

(0) $\quad I(f \wedge t) \downarrow 0$ for $t \downarrow 0$ and $I(f \wedge t) \uparrow I(f)$ for $t \uparrow \infty$ for all $f \in E$.

But the assumption that I is downward σ continuous does not enforce that it admits measure representations, except in case that E is primitive where this follows from the traditional Daniell-Stone theorem. All this shows that the present notion is too superficial in order to be the central one in our enterprise.

We turn to the true central notion. For $\bullet = \star\sigma\tau$ we define a $\bullet$ **representation** of I to be a representation $\alpha : \mathfrak{A} \rightarrow [0, \infty]$ of I such that α is an inner $\bullet$ extension of $\alpha|\mathfrak{T}(E)$. This time the word *inner* is redundant, because there will be no outer counterpart. Our aim is to characterize those I which admit $\bullet$ representations, and then to describe all $\bullet$ representations of I.

We start to define the crude outer and inner envelopes $I^\star, I_\star : [0, \infty]^X \rightarrow [0, \infty]$ of I to be

$$\begin{aligned} I^\star(f) &= \inf\{I(u) : u \in E \text{ with } u \geqq f\}, \\ I_\star(f) &= \sup\{I(u) : u \in E \text{ with } u \leqq f\}. \end{aligned}$$

These envelopes induce set functions $\Delta, \nabla : \mathfrak{T}(E) \rightarrow [0, \infty[$, defined to be

$$\Delta(A) = I^\star(\chi_A) \text{ and } \nabla(A) = I_\star(\chi_A) \quad \text{for } A \in \mathfrak{T}(E).$$

Of course $I_\star \leqq I^\star$ and $\nabla \leqq \Delta$. One proves the criterion which follows.

PROPOSITION. *Assume that I fulfils* (0). *A content $\alpha : \mathfrak{A} \rightarrow [0, \infty]$ on a ring $\mathfrak{A}$ which contains $\mathfrak{T}(E)$ is a representation of I iff $\nabla \leqq \alpha|\mathfrak{T}(E) \leqq \Delta$. If furthermore $\alpha|\mathfrak{T}(E)$ is downward σ continuous then $\alpha|\mathfrak{T}(E) = \Delta$.*

This makes clear that the cases $\bullet = \sigma\tau$ and $\bullet = \star$ fall apart. In the present introduction we shall restrict ourselves to the case $\bullet = \sigma\tau$, which is the simpler and the more important one. From the former main theorem we obtain at once what follows.

CONSEQUENCE (for $\bullet = \sigma\tau$). *I admits $\bullet$ representations iff it fulfils* (0) *and Δ is an inner $\bullet$ premeasure. Then the $\bullet$ representations of I are the inner $\bullet$ extensions of Δ. In particular I has the unique maximal $\bullet$ representation $\Delta_\bullet|\mathfrak{C}(\Delta_\bullet)$.*

This is not yet the desired characterization, because it is not in terms of I itself. In order to achieve this we form for $\bullet = \sigma\tau$ the precise counterparts $I_\bullet : [0,\infty]^X \to [0,\infty]$ of the previous inner $\bullet$ envelopes, that is

$$I_\sigma(f) = \sup\left\{\lim_{l\to\infty} I(u_l) : (u_l)_l \text{ in } E \text{ with } u_l \downarrow \text{some function} \leqq f\right\},$$

and the respective $I_\tau(f)$. We also form for $v \in E$ the satellites $I_\bullet^v : [0,\infty]^X \to [0,\infty[$ in the same sense as before. In these terms our main theorem then reads as follows.

THEOREM (for $\bullet = \sigma\tau$). *For an elementary integral $I : E \to [0,\infty[$ the following are equivalent.*

1) *I admits $\bullet$ representations.*

2) *$I(v) = I(u) + I_\bullet(v-u)$ for all $u \leqq v$ in E.*

3) *$I_\bullet|E = I$; and $I(v) \leqq I(u) + I_\bullet(v-u)$ for all $u \leqq v$ in E.*

3') *$I_\bullet(0) = 0$; and $I(v) \leqq I(u) + I_\bullet^v(v-u)$ for all $u \leqq v$ in E.*

The two last results are the precise counterpart of the main theorem on measure extensions for $\bullet = \sigma\tau$. It is our substitute for the traditional Daniell-Stone theorem. We note that

$$I_\star \leqq I_\sigma \leqq I_\tau, \text{ and } I_\star(f) = I_\star^v(f) \leqq I_\sigma^v(f) \leqq I_\tau^v(f) \quad \text{for } 0 \leqq f \leqq v \in E.$$

Also $I_\star|E = I$, and for $\bullet = \sigma\tau$ the equivalents to $I_\bullet|E = I$ and $I_\bullet(0) = 0$ are as before. We define I to be $\bullet$ **tight** iff it fulfils the second partial condition in 3'). The former crude envelope $I_\star$ allows to define I to be $\star$ **tight** iff

$$I(v) \leqq I(u) + I_\star(v-u) \quad \text{for all } u \leqq v \text{ in } E.$$

An earlier result due to Topsøe [1976] after Pollard-Topsøe [1975] was that 3') with $\star$ tight instead of $\bullet$ tight implies 1). But the converse is not true.

In order to obtain the traditional Daniell-Stone theorem we assume for a moment that E is primitive. Then each I is $\star$ tight and hence $\bullet$ tight. Thus I *admits $\bullet$ representations iff* $I_\bullet(0) = 0$. In this case it has the unique maximal $\bullet$ representation $\Delta_\bullet|\mathfrak{C}(\Delta_\bullet)$, which in particular is a measure representation of I. Thus we obtain for $\bullet = \sigma$ much more than the nontrivial direction in the traditional Daniell-Stone theorem.

We next attempt to incorporate the Riesz representation theorem. We assume X to be a Hausdorff topological space. Let $E \subset [0,\infty[^X$ be a Stonean lattice cone. We need certain conditions on E in order to relate E to the compact subsets of X. One assumption is that E be **concentrated on compacts**, defined to mean that $\mathfrak{T}(E) \subset \mathrm{Comp}(X)$. It implies that E is contained in the class $\mathrm{USC}^+(X)$ of $[0,\infty[$-valued upper semicontinuous functions on X, and that its members are bounded. On the other hand, when E is contained in the subclass

$$\mathrm{USCK}^+(X) := \{f \in \mathrm{USC}^+(X) : f = 0 \text{ outside of some } K \in \mathrm{Comp}(X)\},$$

then E is of course concentrated on compacts. The other assumption is that E be **rich**, defined to mean that

$$\chi_K = \inf\{f \in E : f \geqq \chi_K\} \quad \text{for all } K \in \mathrm{Comp}(X).$$

To see the relevance of this condition note that $\mathrm{CK}^+(X,\mathbb{R})$ is rich iff X is locally compact.

Then the $\bullet = \tau$ version of our Daniell-Stone theorem, combined with the classical Dini theorem, has as an almost immediate consequence the Riesz type theorem which follows.

THEOREM. *Assume that the Stonean lattice cone E is concentrated on compacts and rich. For an elementary integral $I : E \to [0,\infty[$ then the following are equivalent.*

0) *I admits a Radon measure representation* (note that $\mathfrak{T}(E) \subset \mathrm{Comp}(X) \subset \mathrm{Bor}(X)$).

1) *I admits τ representations.*

2) *$I(v) = I(u) + I_\tau(v-u)$ for all $u \leqq v$ in E.*

3') *$I(f \wedge t) \downarrow 0$ for $t \downarrow 0$ for all $f \in E$* (this is redundant when $E \subset \mathrm{USCK}^+(X)$)*; and I is τ tight.*

In this case I has the unique Radon measure representation $\Delta_\tau|\mathrm{Bor}(X)$ with $\mathrm{Bor}(X) \subset \mathfrak{C}(\Delta_\tau)$, which therefore is a τ representation of I.

Let us look at the particular case $E \subset \mathrm{USCK}^+(X)$. Then each Radon measure $\alpha : \mathrm{Bor}(X) \to [0,\infty]$ defines an elementary integral I on E via $I(f) = \int f d\alpha$ for $f \in E$. This I is τ tight in view of 0)⇒3'). Thus we obtain what follows.

THEOREM. *Assume that the Stonean lattice cone $E \subset \mathrm{USCK}^+(X)$ is rich. Then there is a one-to-one correspondence between the elementary integrals $I : E \to [0,\infty[$ which are τ tight and the Radon measures $\alpha : \mathrm{Bor}(X) \to [0,\infty]$. The correspondence is $I(f) = \int f d\alpha$ for all $f \in E$.*

It seems that this is the first Riesz representation theorem which applies to all Hausdorff topological spaces X and contains the traditional Riesz theorem as a direct specialization. In fact, if E is primitive then each I is $\star$ tight and hence τ tight, as we have seen above. Thus for locally compact X and $E = \mathrm{CK}^+(X,\mathbb{R})$ we obtain the the traditional Riesz theorem.