Graduate Texts in Mathematics

58

Artist's conception of the 3-adic unit disk.

Drawing by A.T. Fomenko of Moscow State University, Moscow, U.S.S.R.

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p-adic Numbers, *p*-adic Analysis, and Zeta-Functions

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To Professor Mark Kac

Preface

These lecture notes are intended as an introduction to p-adic analysis on the elementary level. For this reason they presuppose as little background as possible. Besides about three semesters of calculus, I presume some slight exposure to more abstract mathematics, to the extent that the student won't have an adverse reaction to matrices with entries in a field other than the real numbers, field extensions of the rational numbers, or the notion of a continuous map of topological spaces.

The purpose of this book is twofold: to develop some basic ideas of p-adic analysis, and to present two striking applications which, it is hoped, can be as effective pedagogically as they were historically in stimulating interest in the field. The first of these applications is presented in Chapter II, since it only requires the most elementary properties of $\mathbb{Q}_p$; this is Mazur's construction by means of p-adic integration of the Kubota–Leopoldt p-adic zeta-function, which "p-adically interpolates" the values of the Riemann zeta-function at the negative odd integers. My treatment is based on Mazur's Bourbaki notes (unpublished). The book then returns to the foundations of the subject, proving extension of the p-adic absolute value to algebraic extensions of $\mathbb{Q}_p$, constructing the p-adic analogue of the complex numbers, and developing the theory of p-adic power series. The treatment highlights analogies and contrasts with the familiar concepts and examples from calculus. The second main application, in Chapter V, is Dwork's proof of the rationality of the zeta-function of a system of equations over a finite field, one of the parts of the celebrated Weil Conjectures. Here the presentation follows Serre's exposition in *Séminaire Bourbaki*.

These notes have no pretension to being a thorough introduction to p-adic analysis. Such topics as the Hasse–Minkowski Theorem (which is in Chapter 1 of Borevich and Shafarevich's *Number Theory*) and Tate's thesis (which is also available in textbook form, see Lang's *Algebraic Number Theory*) are omitted.

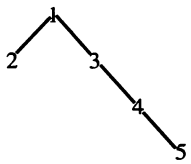
Moreover, there is no attempt to present results in their most general form. For example, p-adic L-functions corresponding to Dirichlet characters are only discussed parenthetically in Chapter II. The aim is to present a selection of material that can be digested by undergraduates or beginning graduate students in a one-term course.

The exercises are for the most part not hard, and are important in order to convert a passive understanding to a real grasp of the material. The abundance of exercises will enable many students to study the subject on their own, with minimal guidance, testing themselves and solidifying their understanding by working the problems.

p-adic analysis can be of interest to students for several reasons. First of all, in many areas of mathematical research—such as number theory and representation theory—p-adic techniques occupy an important place. More naively, for a student who has just learned calculus, the "brave new world" of non-Archimedean analysis provides an amusing perspective on the world of classical analysis. p-adic analysis, with a foot in classical analysis and a foot in algebra and number theory, provides a valuable point of view for a student interested in any of those areas.

I would like to thank Professors Mark Kac and Yu. I. Manin for their help and encouragement over the years, and for providing, through their teaching and writing, models of pedagogical insight which their students can try to emulate.

Logical dependence of chapters



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