Anthony W. Knapp

Basic Algebra

Along with a companion volume *Advanced Algebra*

Birkhäuser Boston • Basel • Berlin Anthony W. Knapp 81 Upper Sheep Pasture Road East Setauket, NY 11733-1729 U.S.A. e-mail to: aknapp@math.sunysb.edu http://www.math.sunysb.edu/~aknapp/books/b-alg.html

Cover design by Mary Burgess.

Mathematics Subject Classicification (2000): 15-01, 20-02, 13-01, 12-01, 16-01, 08-01, 18A05, 68P30

Library of Congress Control Number: 2006932456

ISBN-10 0-8176-3248-4	eISBN-10 0-8176-4529-2
ISBN-13 978-0-8176-3248-9	eISBN-13 978-0-8176-4529-8
Advanced Algebra	ISBN 0-8176-4522-5
Basic Algebra and Advanced Algeb	<i>ISBN 0-8176-4533-0</i> ISBN 0-8176-4533-0

Printed on acid-free paper.

©2006 Anthony W. Knapp

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Birkhäuser Boston, c/o Springer Science+Business Media LLC, 233 Spring Street, New York, NY 10013, USA) and the author, except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use in this publication of trade names, trademarks, service marks and similar terms, even if they are not identified as such, is not to be taken as an expression of opinion as to whether or not they are subject to proprietary rights.

987654321

www.birkhauser.com

CONTENTS

	Con	atents of Advanced Algebra	Х		
	List of Figures				
	Preface Dependence Among Chapters				
	Standard Notation				
	Gui	de for the Reader	xix		
I.		ELIMINARIES ABOUT THE INTEGERS,			
	POI	LYNOMIALS, AND MATRICES	1		
	1.	Division and Euclidean Algorithms	1		
	2.	Unique Factorization of Integers	4		
	3.	Unique Factorization of Polynomials	9		
		Permutations and Their Signs	15		
		Row Reduction	19		
		Matrix Operations	24		
	7.	Problems	30		
II.	VE	CTOR SPACES OVER $\mathbb{Q}, \mathbb{R},$ AND \mathbb{C}	33		
	1.	Spanning, Linear Independence, and Bases	33		
	2.	Vector Spaces Defined by Matrices	38		
	3.	Linear Maps	42		
	4.	Dual Spaces	50		
		Quotients of Vector Spaces	54		
	6.	Direct Sums and Direct Products of Vector Spaces	58		
	7.	Determinants	65		
	8.	Eigenvectors and Characteristic Polynomials	73		
	9.	Bases in the Infinite-Dimensional Case	77		
	10.	Problems	82		
III.	INN	IER-PRODUCT SPACES	88		
	1.	Inner Products and Orthonormal Sets	88		
	2.	Adjoints	98		
	3.	Spectral Theorem	104		
	4.	Problems	111		

viii		Contents	
IV.	GR	OUPS AND GROUP ACTIONS	116
	1.	Groups and Subgroups	117
	2.	Quotient Spaces and Homomorphisms	128
	3.	Direct Products and Direct Sums	134
	4.	Rings and Fields	140
	5.	Polynomials and Vector Spaces	147
	6.		158
	7.	Semidirect Products	166
	8.	Simple Groups and Composition Series	170
	9.	Structure of Finitely Generated Abelian Groups	174
	10.	Sylow Theorems	183
	11.	Categories and Functors	188
	12.	Problems	198
V.	TH	EORY OF A SINGLE LINEAR TRANSFORMATION	209
	1.	Introduction	209
	2.	Determinants over Commutative Rings with Identity	212
	3.	Characteristic and Minimal Polynomials	216
	4.	Projection Operators	224
	5.	Primary Decomposition	226
	6.		229
		Computations with Jordan Form	235
	8.	Problems	239
VI.	MU	LTILINEAR ALGEBRA	245
	1.	Bilinear Forms and Matrices	246
	2.	Symmetric Bilinear Forms	250
	3.	Alternating Bilinear Forms	253
	4.	Hermitian Forms	255
	5.	I U	257
	6.	I	260
	7.	Tensor Algebra	274
	8.	Symmetric Algebra	280
		Exterior Algebra	288
	10.	Problems	292
VII.	AD	VANCED GROUP THEORY	303
	1.	Free Groups	303
	2.	Subgroups of Free Groups	314
	3.	Free Products	319
	4.	Group Representations	326

VII. ADVANCED GROUP THEORY (Continued)5. Burnside's Theorem3426. Extensions of Groups3447. Problems357VIII. COMMUTATIVE RINGS AND THEIR MODULES3671. Examples of Rings and Modules3672. Integral Domains and Fields of Fractions3783. Prime and Maximal Ideals3814. Unique Factorization3845. Gauss's Lemma3906. Finitely Generated Modules3967. Orientation for Algebraic Number Theory and Algebraic Geometry4088. Noetherian Rings and the Hilbert Basis Theorem4149. Integral Closure41710. Localization and Local Rings42511. Dedekind Domains43412. Problems439
6. Extensions of Groups3447. Problems357VIII. COMMUTATIVE RINGS AND THEIR MODULES1. Examples of Rings and Modules3672. Integral Domains and Fields of Fractions3783. Prime and Maximal Ideals3814. Unique Factorization3845. Gauss's Lemma3906. Finitely Generated Modules3967. Orientation for Algebraic Number Theory and Algebraic Geometry4088. Noetherian Rings and the Hilbert Basis Theorem4149. Integral Closure41710. Localization and Local Rings42511. Dedekind Domains434
6. Extensions of Groups3447. Problems357VIII. COMMUTATIVE RINGS AND THEIR MODULES1. Examples of Rings and Modules3672. Integral Domains and Fields of Fractions3783. Prime and Maximal Ideals3814. Unique Factorization3845. Gauss's Lemma3906. Finitely Generated Modules3967. Orientation for Algebraic Number Theory and Algebraic Geometry4088. Noetherian Rings and the Hilbert Basis Theorem4149. Integral Closure41710. Localization and Local Rings42511. Dedekind Domains434
7. Problems357VIII. COMMUTATIVE RINGS AND THEIR MODULES3671. Examples of Rings and Modules3672. Integral Domains and Fields of Fractions3783. Prime and Maximal Ideals3814. Unique Factorization3845. Gauss's Lemma3906. Finitely Generated Modules3967. Orientation for Algebraic Number Theory and Algebraic Geometry4088. Noetherian Rings and the Hilbert Basis Theorem4149. Integral Closure41710. Localization and Local Rings42511. Dedekind Domains434
1.Examples of Rings and Modules3672.Integral Domains and Fields of Fractions3783.Prime and Maximal Ideals3814.Unique Factorization3845.Gauss's Lemma3906.Finitely Generated Modules3967.Orientation for Algebraic Number Theory and Algebraic Geometry4088.Noetherian Rings and the Hilbert Basis Theorem4149.Integral Closure41710.Localization and Local Rings42511.Dedekind Domains434
2.Integral Domains and Fields of Fractions3783.Prime and Maximal Ideals3814.Unique Factorization3845.Gauss's Lemma3906.Finitely Generated Modules3967.Orientation for Algebraic Number Theory and Algebraic Geometry4088.Noetherian Rings and the Hilbert Basis Theorem4149.Integral Closure41710.Localization and Local Rings42511.Dedekind Domains434
3. Prime and Maximal Ideals3814. Unique Factorization3845. Gauss's Lemma3906. Finitely Generated Modules3967. Orientation for Algebraic Number Theory and Algebraic Geometry4088. Noetherian Rings and the Hilbert Basis Theorem4149. Integral Closure41710. Localization and Local Rings42511. Dedekind Domains434
4.Unique Factorization3845.Gauss's Lemma3906.Finitely Generated Modules3967.Orientation for Algebraic Number Theory and Algebraic Geometry4088.Noetherian Rings and the Hilbert Basis Theorem4149.Integral Closure41710.Localization and Local Rings42511.Dedekind Domains434
5. Gauss's Lemma3906. Finitely Generated Modules3967. Orientation for Algebraic Number Theory and Algebraic Geometry4088. Noetherian Rings and the Hilbert Basis Theorem4149. Integral Closure41710. Localization and Local Rings42511. Dedekind Domains434
5. Gauss's Lemma3906. Finitely Generated Modules3967. Orientation for Algebraic Number Theory and Algebraic Geometry4088. Noetherian Rings and the Hilbert Basis Theorem4149. Integral Closure41710. Localization and Local Rings42511. Dedekind Domains434
7. Orientation for Algebraic Number Theory and Algebraic Geometry4088. Noetherian Rings and the Hilbert Basis Theorem4149. Integral Closure41710. Localization and Local Rings42511. Dedekind Domains434
Algebraic Geometry4088. Noetherian Rings and the Hilbert Basis Theorem4149. Integral Closure41710. Localization and Local Rings42511. Dedekind Domains434
Algebraic Geometry4088. Noetherian Rings and the Hilbert Basis Theorem4149. Integral Closure41710. Localization and Local Rings42511. Dedekind Domains434
8. Noetherian Rings and the Hilbert Basis Theorem4149. Integral Closure41710. Localization and Local Rings42511. Dedekind Domains434
9. Integral Closure41710. Localization and Local Rings42511. Dedekind Domains434
10. Localization and Local Rings42511. Dedekind Domains434
11. Dedekind Domains 434
12. Problems439
IX. FIELDS AND GALOIS THEORY 448
1. Algebraic Elements 449
2. Construction of Field Extensions 453
3. Finite Fields 457
4. Algebraic Closure 460
5. Geometric Constructions by Straightedge and Compass 464
6. Separable Extensions 469
7. Normal Extensions 476
8. Fundamental Theorem of Galois Theory 479
9. Application to Constructibility of Regular Polygons 483
10. Application to Proving the Fundamental Theorem of Algebra 486
11. Application to Unsolvability of Polynomial Equations with
Nonsolvable Galois Group 488
12. Construction of Regular Polygons 493
13. Solution of Certain Polynomial Equations with Solvable
Galois Group 501
14. Proof That π Is Transcendental 510
15. Norm and Trace 514
16.Splitting of Prime Ideals in Extensions521
13.Spinning of Finite Teens in Entensions52717.Two Tools for Computing Galois Groups527
18. Problems 534

Contents

X.	MO	DULES OVER NONCOMMUTATIVE RINGS	544
	1.	Simple and Semisimple Modules	544
	2.	Composition Series	551
	3.	Chain Conditions	556
	4.	Hom and End for Modules	558
	5.	Tensor Product for Modules	565
	6.	Exact Sequences	574
	7.	Problems	579
AP	APPENDIX		583
	A1.	Sets and Functions	583
	A2.	Equivalence Relations	589
	A3.	Real Numbers	591
	A4.	Complex Numbers	594
		Partial Orderings and Zorn's Lemma	595
	A6.	Cardinality	599
	Hint	ts for Solutions of Problems	603
	Selected References		
		x of Notation	699
	Index		