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Group Methods in Commutative Harmonic Analysis

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Preface to the English Edition

This book is aimed at a wide circle of readers, including mathematicians, theoretical physicists, and engineers. It contains a group theoretic treatment of the classical problems of harmonic analysis: eigenfunctions of the Fourier operator, the Fourier transform in the complex domain and related transforms (the Laplace, Hilbert, Borel and Carleman transforms), shift-invariant subspaces, unitary representations of groups, positive definite functions, the arithmetic of characteristic functions of probability measures, the completeness and compactness of systems of translations, Tauberian theory, and problems of spectral analysis and spectral synthesis. The main notions, ideas and facts of invariant integration and duality, as related to the Fourier transform on groups, are illustrated by numerous examples, as are the required group theoretic methods.

A great number of themes of modern mathematics is discussed under the title "Harmonic Analysis." (cf. the introductory Volume 15 of the present series). The range of these themes is so remarkably wide that a specialist working on one of them may well be unaware of the terminology used by his colleague working on another (and vice versa), even though both of them are sure that they work on harmonic analysis or its applications. One of the main incentives for the author to write this volume was therefore to bring together as many different branches of commutative harmonic analysis as possible in order to emphasize their interactions.

At the turn of the nineteenth century, two major events happened in mathematics whose impact cannot be overestimated. On the one hand, Fourier had, as Riemann put it, correctly explained the nature of trigonometric series, and, on the other, Gauss made use of numerical characters in arithmetic in a systematic way, thus enriching it with new tools, discoveries and applications. These two disciplines looked so different that their common algebraic (group theoretic) nature was understood only in the twentieth century.

Meanwhile, half a century after Fourier and Gauss, Riemann gave the first rigorous foundation of the notion of the integral in his thesis devoted to trigonometric series. This circumstance had almost a symbolic nature: from that moment on, the notions of the integral and the trigonometric series became inseparable. The desire to pursue these connections was another important incentive for writing this book.

The author is aware of the fact that his understanding of the main themes of harmonic analysis, his choices and approach are necessarily subjective. Nevertheless, he hopes that this survey will be of interest to a wide audience. It is written for the general reader and yet may have some points that may prove stimulating to the specialist.

The present edition is a thoroughly revised, corrected and expanded version of the original Russian edition. Many factors motivated this choice. The Russian edition was published more than seven years ago, during which time many new important results (and even areas) worthy of mention appeared. The author had the possibility of discussing the topics presented below with numerous mathematicians working in this field and so became aware of many publications that are of direct interest for the subject of this volume. As a result the list of references was substantially expanded. And there was another reason. This volume was conceived as an introductory volume to a subsequent, more specialized publication, devoted to the spectral theory of functions and its different applications. As the latter book has had to be indefinitely postponed, the author decided to add some of its topics to this volume.

The author is very grateful to V. P. Havin for his invaluable advices, discussions and support.

The author's stimulating discussions with V. Katsnel'son were extremely fruitful and helped to improve the text. Special thanks are due to H. Helson of the University of California (Berkeley) and to K. Ross of the University of Oregon for useful discussions. The author wishes to express his gratitude to the Weizmann Institute of Sciences in Israel for giving him the opportunity to complete the work on this English edition. The author is grateful also to RFBR (grant 95-01-00965a) for its support.

Finally, the author is pleased to thank the translators of this volume, Dr. S. Dynin and Mr. D. Dynin, for their patient and attentive collaboration.

January, 1998 *V. P. Gurarii*

Group Methods in Commutative Harmonic Analysis

V. P. Gurarii

Translated from the Russian
by D. and S. Dynin

Contents

Preface

For a long time now motions with periodic properties have been viewed as summations of simpler harmonic motions. The problem of retrieving the pure harmonics hidden in a complicated motion was called a harmonic analysis problem, while the problem of reconstructing a complicated motion from these harmonics became known as a harmonic synthesis problem.

Around the turn of the 19th century pure harmonics were found to satisfy the equation $f(x+y) = f(x)f(y)$, x, $y \in \mathbb{R}$, which indicated that certain group principles lie at the foundation of harmonic analysis. Approximately at the same time appeared the so-called multiplicative numerical functions in arithmetic. They satisfied the equation $f(m \cdot n) = f(m) \cdot f(n)$ for relatively prime m and n; the study of such functions on the residue ring $\bmod m$ led to the definition of numerical characters $\bmod m$. If a numerical character χ is restricted to the group G_m^* of invertible elements of the residue ring $\bmod m$, the relation $\chi(a \cdot b) = \chi(a)\chi(b)$, $a, b \in G_m^*$, becomes analogous to the equation for pure harmonics. This analogy is not a coincidence.

This implicit group point of view has accompanied harmonic analysis (and number theory) along the many stages of their more than 200-year-long history.

The decisive breakthrough came as the result (well prepared by previous research) of a rapid succession of papers falling within just a single decade: from the mid-twenties to the mid-thirties of this century.

The rapid development of quantum mechanics stimulated research in operator theory and group representation theory. Initiated during the mid-twenties, intensive study of topological groups and their representations led to Haar's discovery of the basic construction of invariant integration on a topological group. Bohr's theory of almost periodic functions influenced the work of Wiener, Bochner and many other analysts. They enriched the technical arsenal of harmonic analysis and the scope of its applications (statistical mechanics, ergodic theory, time series, etc.) The new notion of the generalized Fourier transform made it possible to consider Plancherel's theory simultaneously with Bohr's theory, the continuous spectrum with the discrete. The Pontrjagin-van Kampen duality opened the way for an unobstructed development of Fourier analysis on locally compact abelian groups, allowing Fourier series, Fourier integrals and expansions via numerical characters to be viewed as objects of the same kind. The Peter-Weyl theory made it possible for von Neumann to analyze almost periodic functions on groups by connecting them to group representation theory. Along with the many other discoveries of that period, this led to the inclusion of group theorethical methods into the tool kit of harmonic analysis.

The intensive studies which followed resulted in the development of a vast field of abstract harmonic analysis for which the use of group theoretical methods is especially characteristic.

The basic principles which underpin these methods, along with the history of their origins and their impact on the development of many mathematical fields from mathematical physics to arithmetic, are described in Mackey's captivating survey (cf. Mackey (1978)), to which we will refer many times in our exposition.

Let us list some research directions in modern harmonic analysis: positive definite functions and kernels; almost periodic functions and representations; spectral function theory and its central problem of spectral synthesis; functions on groups periodic in the mean and convolution equations; harmonic analysis on totally disconnected groups and nontrigonometric Fourier series (Walsh-type series); harmonic analysis on local fields and on the adele ring and its applications in number theory and representation theory; the mysterious Banach algebra of bounded Borel measures on locally compact abelian groups and its hidden "surprises," foremost among which is the so-called Wiener-Pitt "hidden spectrum" effect; Tauberian theory and harmonic analysis; translation-invariant operators and invariant subspaces; cardinal series, the Sampling Theorem and its role in signal theory; wavelets; and the list can be continued. All of these areas are rapidly developing, and progress in them cannot avoid the group point of view.

The limited space of this book prevents us from giving even a superfluous survey of the subjects mentioned. Nevertheless, we have tried to give the reader an idea of them and of the general principles behind them, all while at the same time trying to retain some unity of exposition, and to this end we have divided the book into two independent parts.

The first part (Chapter 1) is devoted to the problems of harmonic analysis on the real axis $\mathbb{R}$. Their set-up is mostly dictated by the group nature of $\mathbb{R}$ (they can therefore be considered in a more general group situation). Nevertheless, their solution requires various methods of function theory. Sometimes these methods seem to be absolutely necessary. At other times one may hope to either transform them in such a way as to make them applicable in a more general group situation or to find an alternative group approach. There are many interesting books devoted to classical Fourier analysis and its methods, such as the elegant book of H. Helson (Helson (1983)), two monographs of H. Dym and H. P. McKean (Dym, McKean (1972) and (1976)) which are very close to our Chapter 1, T.W. Körner's very rich book which has numerous examples and applications (Körner (1988)), G. B. Folland's book (Folland (1989)) which opens new areas and new horizons of the modern harmonic analysis, as well as the books of P. Koosis (Koosis (1988)) and V. Havin and B. Joericke (Havin, Joericke (1994)), which contain methods and tools that may be of help in the study of different problems of harmonic analysis, and many others (we will refer to them in the text).

In the second part (Chapter 2) we deal with the basic notions and facts lying at the foundation of modern abstract harmonic analysis. The contents of this chapter are illustrated with numerous examples, which provide an approach to some of the themes listed above.

Despite the independence of the two chapters, the intersection of their contents is nonempty. Therefore, if we deal freely with a notion or definition in one of the chapters without explaining it, then the corresponding explanation should be sought in the other chapter, using, for example, the subject index.

Since we will often refer, especially in Chapter 2, to the encyclopaedic monograph of Hewitt and Ross (cf. Hewitt, Ross (1963) and (1970)), we will denote it by the abbreviation HR. Furthemore, we will use throughout the book the standard abbreviation LCA group for a locally compact abelian group.

A reference to an article from the series Encyclopaedia of Mathematical Sciences will be indicated only by the author's name and the volume number.

We want to say a few words about the organization of material in this book, the cross-references to equations, theorems, definitions, etc. Each chapter consists of sections, and each section consists of subsections. Each subsection has its own numeration of equations and definitions; for example, a reference to Equation (2) inside of a subsection indicates Equation (2) of the same subsection. A reference to an equation outside of a subsection is given as follows: for example, (8.4.1) refers to Equation (1) from Subsection 4 of Section 8. Similarly, Corollary 8.1.4 is to be found in Section 8, Subsection 1, under number 4. Finally, Proposition 2.1.(7) will be found under number (7), in Subsection 1 of Section 2.

The reader should be warned that "proofs"given in the text are rather "sketches of proof". However we hope they are sufficiently suggestive for any interested mathematician to recover all missing details.

In conclusion, I express my gratitude to N. K. Nikolskii, who read the preliminary manuscript of the book and made a series of remarks, which I tried to take into account. I am also grateful to Yu. I. Lyubarskij who read the final version of the manuscript and corrected some of its inaccuracies.

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