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Measure and Integral

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PREFACE

This is a systematic exposition of the basic part of the theory of measure and integration. The book is intended to be a usable text for students with no previous knowledge of measure theory or Lebesgue integration, but it is also intended to include the results most commonly used in functional analysis. Our two intentions are some what conflicting, and we have attempted a resolution as follows.

The main body of the text requires only a first course in analysis as background. It is a study of abstract measures and integrals, and comprises a reasonably complete account of Borel measures and integration for $\mathbb{R}$. Each chapter is generally followed by one or more supplements. These, comprising over a third of the book, require somewhat more mathematical background and maturity than the body of the text (in particular, some knowledge of general topology is assumed) and the presentation is a little more brisk and informal. The material presented includes the theory of Borel measures and integration for $\mathbb{R}^n$, the general theory of integration for locally compact Hausdorff spaces, and the first dozen results about invariant measures for groups.

Most of the results expounded here are conventional in general character, if not in detail, but the methods are less so. The following brief overview may clarify this assertion.

The first chapter prepares for the study of Borel measures for $\mathbb{R}$. This class of measures is important and interesting in its own right and it furnishes nice illustrations for the general theory as it develops. We begin with a brief analysis of length functions, which are functions on the class $\mathscr{J}$ of closed intervals that satisfy three axioms which are eventually shown to ensure that they extend to measures. It is shown in chapter 1 that every length function has a unique extension λ to the lattice $\mathscr{L}$ of sets generated by $\mathscr{J}$ so that λ is *exact*, in the sense that $\lambda(A) = \lambda(B) + sup\{\lambda(C): C \in \mathscr{L} \text{ and } C \subset A \setminus B\}$ for members A and B of $\mathscr{L}$ with $A \subset B$.

The second chapter details the construction of a pre-integral from a pre-measure. A real valued function μ on a family $\mathscr{A}$ of sets that is closed under finite intersection is a *pre-measure* iff it has a countably additive non-negative extension to the ring of sets generated by $\mathscr{A}$ (e.g., an exact function μ that is continuous at $\varnothing$). Each length function is a pre-measure. If μ is an exact function on $\mathscr{A}$, the map $\chi_A \mapsto \mu(A)$ for A in $\mathscr{A}$ has a linear extension I to the vector space L spanned by the characteristic functions χ_A, and the space L is a vector lattice with truncation: $I \wedge f \in L$ if $f \in L$. If μ is a pre-measure, then the positive linear functional I has the property: if $\{f_n\}_n$ is a decreasing sequence in L that converges pointwise to zero, then $lim_n I(f_n) = 0$. Such a functional I is a *pre-integral*. An *integral* is a pre-integral with the Beppo Levi property: if $\{f_n\}_n$ is an increasing sequence in L converging pointwise to a function f and $sup_n I(f_n) < \infty$, then $f \in L$ and $lim_n I(f_n) = I(f)$.

In chapter 3 we construct the Daniell–Stone extension L^1 of a pre-integral I on L by a simple process which makes clear that the extension is a completion under the L^1 norm $\|f\|_1 = I(|f|)$. Briefly: a set E is called *null* iff there is a sequence $\{f_n\}_n$ in L with $\sum_n \|f_n\|_1 < \infty$ such that $\sum_n |f_n(x)| = \infty$ for all x in E, and a function g belongs to L^1 iff g is the pointwise limit, except for the points in some null set, of a sequence $\{g_n\}_n$ in L such that $\sum_n \|g_{n+1} - g_n\|_1 < \infty$ (such sequences are called *swiftly convergent*). Then L^1 is a norm completion of L and the natural extension of I to L^1 is an integral. The methods of the chapter, also imply for an arbitrary integral, that the domain is norm complete and the monotone convergence and the dominated convergence theorems hold. These results require no measure theory; they bring out vividly the fundamental character of M. H. Stone's axioms for an integral.

A *measure* is a real (finite) valued non-negative countably additive function on a δ-ring (a ring closed under countable intersection). If J is an arbitrary integral on M, then the family $\mathscr{A} = \{A: \chi_A \in M\}$ is a δ-ring and the function $A \mapsto J(\chi_A)$ is a measure, the measure induced by the integral J. Chapter 4 details this procedure and applies the result, together with the pre-measure to pre-integral to integral theorems of the preceding chapters to show that each exact function that is continuous at $\varnothing$ has an extension that is a measure. A supplement presents the standard construction of regular Borel measures and another supplement derives the existence of Haar measure.

A measure μ on a δ-ring $\mathscr{A}$ is also a pre-measure; it induces a pre-integral, and this in turn induces an integral. But there is a more direct way to obtain an integral from the measure μ: A real valued function f belongs to $L_1(\mu)$ iff there is $\{a_n\}_n$ in $\mathbb{R}$ and $\{A_n\}_n$ in $\mathscr{A}$ such that

$\sum_n |a_n|\mu(A_n) < \infty$ and $f(x) = \sum_n a_n \chi_{A_n}(x)$ for all x, and in this case the *integral* $I_\mu(f)$ is defined to be $\sum_n a_n \mu(A_n)$. This construction is given in chapter 6, and it is shown that every integral is the integral with respect to the measure it induces.

Chapter 6 requires facts about measurability that are purely set theoretic in character and these are developed in chapter 5. The critical results are: Call a function f $\mathscr{A}$ *σ-simple* (or $\mathscr{A}$ *σ^+-simple*) iff $f = \sum_n a_n \chi_{A_n}$ for some $\{A_n\}_n$ in $\mathscr{A}$ and $\{a_n\}_n$ in $\mathbb{R}$ (in $\mathbb{R}^+$, respectively). Then, if $\mathscr{A}$ is a δ-ring, a real valued function f is $\mathscr{A}$ σ-simple iff it has a support in $\mathscr{A}_\sigma$ and is locally $\mathscr{A}$ measurable (if B is an arbitrary Borel subset of $\mathbb{R}$, then $A \cap f^{-1}[B]$ belongs to $\mathscr{A}$ for each A in $\mathscr{A}$). Moreover, if such a function is non-negative, it is $\mathscr{A}$ σ^+-simple.

Chapter 7 is devoted to product measures and product integrals. It is concerned with conditions that relate the integral of a function f w.r.t. $\mu \otimes v$ to the iterated integrals $\int(\int f(x,y)\, d\mu x)\, dvy$ and $\int(\int f(x,y)\, dvy)\, d\mu x$. We follow the natural approach, deriving the Fubini theorem from the Tonelli theorem, and the latter leads us to grudgingly allow that some perfectly respectable σ-simple functions have infinite integrals (we call these functions *integrable in the extended sense*, or *integrable**).

Countably additive non-negative functions μ to the extended set $\mathbb{R}^*$ of reals (*measures in the extended sense* or *measures**) also arise naturally (chapter 8) as images of measures under reasonable mappings. If μ is a measure on a σ-field $\mathscr{A}$ of subsets of X, $\mathscr{B}$ is a σ-field for Y, and $T: X \to Y$ is $\mathscr{A} - \mathscr{B}$ measurable, then the image measure $T\mu$ is defined by $T\mu(B) = \mu(T^{-1}[B])$ for each B in $\mathscr{B}$. If $\mathscr{A}$ is a δ-ring but not a σ-field, there is a possibly infinite valued measure that can appropriately be called the T image of μ. We compute the image of Borel–Lebesgue measure for $\mathbb{R}$ under a smooth map, and so encounter indefinite integrals.

Indefinite integrals w.r.t. a σ-finite measure μ are characterized in chapter 9, and the principal result, the Radon–Nikodym theorem, is extended to decomposable measures and regular Borel measures in a supplement. Chapter 10 begins the study of Banach spaces. The duals of some standard spaces are characterized, and in a supplement our methods are used to establish very simply, or at least σ-simply, the basic facts about Bochner integrals.

This book is based on various lectures given by one or the other of us in 1965 and later, at the Indian Institute of Technology, Kanpur; Panjab University, Chandigarh; University of California, Berkeley; and the University of Kansas. We were originally motivated by curiosity about how a σ-simple approach would work; it did work, and a version of most of this text appeared as preprints in 1968, 1972 and 1979, under the title "Measures and Integrals." Since that time our point of view has changed on several matters (but not on σ-simplicity) and the techniques have been refined.

This is the first of two volumes on *Measure and Integral*. The exercises, problems, and additional supplements will appear as a companion volume to be published as soon as we can sift and edit a large disorganized mass of manuscript.

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J. L. KELLEY T. P. SRINIVASAN

CONTENTS