

---

# Contents

|                                                                     |    |
|---------------------------------------------------------------------|----|
| Foreword: MASS and REU at Penn State University                     | ix |
| Preface                                                             | xi |
| Chapter 1. Arithmetic of the $p$ -adic Numbers                      | 1  |
| §1.1. From $\mathbb{Q}$ to $\mathbb{R}$ ; the concept of completion | 2  |
| Exercises 1–8                                                       | 5  |
| §1.2. Normed fields                                                 | 6  |
| Exercises 9–16                                                      | 14 |
| §1.3. Construction of the completion of a normed field              | 15 |
| Exercises 17–19                                                     | 19 |
| §1.4. The field of $p$ -adic numbers $\mathbb{Q}_p$                 | 19 |
| Exercises 20–25                                                     | 26 |
| §1.5. Arithmetical operations in $\mathbb{Q}_p$                     | 27 |
| Exercises 26–31                                                     | 30 |
| §1.6. The $p$ -adic expansion of rational numbers                   | 30 |
| Exercises 32–34                                                     | 33 |
| §1.7. Hensel’s Lemma and congruences                                | 33 |
| Exercises 35–44                                                     | 38 |
| §1.8. Algebraic properties of $p$ -adic integers                    | 39 |

---

|                                                                            |     |
|----------------------------------------------------------------------------|-----|
| §1.9. Metrics and norms on the rational numbers.                           |     |
| Ostrowski's Theorem                                                        | 43  |
| Exercises 45–46                                                            | 47  |
| §1.10. A digression: what about $\mathbb{Q}_g$ if $g$ is not a prime?      | 47  |
| Exercises 47–50                                                            | 50  |
| Chapter 2. The Topology of $\mathbb{Q}_p$ vs. the Topology of $\mathbb{R}$ | 53  |
| §2.1. Elementary topological properties                                    | 53  |
| Exercises 51–53                                                            | 60  |
| §2.2. Cantor sets                                                          | 60  |
| Exercises 54–65                                                            | 68  |
| §2.3. Euclidean models of $\mathbb{Z}_p$                                   | 69  |
| Exercises 66–68                                                            | 73  |
| Chapter 3. Elementary Analysis in $\mathbb{Q}_p$                           | 75  |
| §3.1. Sequences and series                                                 | 75  |
| Exercises 69–73                                                            | 80  |
| §3.2. $p$ -adic power series                                               | 80  |
| Exercises 74–78                                                            | 86  |
| §3.3. Can a $p$ -adic power series be analytically continued?              | 87  |
| §3.4. Some elementary functions                                            | 89  |
| Exercises 79–81                                                            | 92  |
| §3.5. Further properties of $p$ -adic exponential and<br>logarithm         | 92  |
| §3.6. Zeros of $p$ -adic power series                                      | 98  |
| Exercises 82–83                                                            | 102 |
| Chapter 4. $p$ -adic Functions                                             | 103 |
| §4.1. Locally constant functions                                           | 103 |
| Exercises 84–87                                                            | 107 |
| §4.2. Continuous and uniformly continuous functions                        | 108 |
| Exercises 88–90                                                            | 112 |
| §4.3. Points of discontinuity and the Baire Category<br>Theorem            | 112 |

|                                                      |     |
|------------------------------------------------------|-----|
| <b>Contents</b>                                      | vii |
| <hr/>                                                |     |
| Exercises 91–96                                      | 115 |
| §4.4. Differentiability of $p$ -adic functions       | 116 |
| §4.5. Isometries of $\mathbb{Q}_p$                   | 121 |
| Exercises 97–100                                     | 123 |
| §4.6. Interpolation                                  | 123 |
| Exercises 101–103                                    | 134 |
| Answers, Hints, and Solutions for Selected Exercises | 135 |
| Bibliography                                         | 149 |
| Index                                                | 151 |