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Methods of Mathematical Finance

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To Eleni, for all her patience.
I.K.

And to Dot, whom I love.
S.E.S.

Preface

This book is intended for readers who are quite familiar with probability and stochastic processes but know little or nothing about finance. It is written in the definition/theorem/proof style of modern mathematics and attempts to explain as much of the finance motivation and terminology as possible.

A mathematical monograph on finance can be written today only because of two revolutions that have taken place on Wall Street in the latter half of the twentieth century. Both these revolutions began at universities, albeit in economics departments and business schools, not in departments of mathematics or statistics. They have led inexorably, however, to an escalation in the level of mathematics (including probability, statistics, partial differential equations and their numerical analysis) used in finance, to a point where genuine research problems in the former fields are now deeply intertwined with the theory and practice of the latter.

The first revolution in finance began with the 1952 publication of "Portfolio Selection," an early version of the doctoral dissertation of Harry Markowitz. This publication began a shift away from the concept of trying to identify the "best" stock for an investor, and towards the concept of trying to understand and quantify the trade-offs between *risk* and *return* inherent in an entire portfolio of stocks. The vehicle for this so-called *mean–variance analysis* of portfolios is *linear regression*; once this analysis is complete, one can then address the *optimization problem* of choosing the portfolio with the largest mean return, subject to keeping the risk (i.e., the variance) below a specified acceptable threshold. The implementation of Markowitz's ideas was aided tremendously by William Sharpe, who developed the concept of determining covariances not between every possible pair of stocks, but between each stock and the "market." For purposes of the above optimization problem each stock could then be characterized by its mean rate of return (its "α") and its correlation with the market (its "β"). For their pioneering work, Markowitz and Sharpe shared with Merton Miller the 1990 Nobel Prize in economics, the first ever awarded for work in finance.

The portfolio-selection work of Markowitz and Sharpe introduced mathematics to the "black art" of investment management. With time, the mathematics has become more sophisticated. Thanks to Robert Merton and Paul Samuelson, one-period models were replaced by continuous-time, Brownian-motion-driven models, and the quadratic utility function implicit in mean–variance optimization was replaced by more general increasing, concave utility functions. Model-based mutual funds have taken a permanent seat at the table of investment opportunities offered to the public. Perhaps more importantly, the paradigm for thinking about financial markets has become a mathematical model. This affects the way we now understand issues of corporate finance, taxation, exchange-rate fluctuations, and all manner of financial issues.

The second revolution in finance is connected with the explosion in the market for *derivative securities.* Already in 1992, this market was estimated by the Bank for International Settlements to be a $4 trillion business worldwide, involving every sector of the finance industry. According to this estimate, the size of the derivative securities market had increased eightfold in five years. The foundational work here was done by Fisher Black, Robert Merton, and Myron Scholes in the early 1970s. Black, Merton, and Scholes were seeking to understand the value of the option to buy one share of a stock at a future *date* and *price* specified in advance. This so-called *European call-option* derives its value from that of the underlying stock, whence the name *derivative security.* The basic idea of valuing a European call-option is to construct a *hedging portfolio*, i.e., a combination of shares from the stock on which the call is written and of shares from a money market, so that the resulting portfolio replicates the option. At any time, the option should be worth exactly as much as the hedging portfolio, for otherwise some astute trader ("arbitrageur") could make something for nothing by trading in the option, the stock, and the money market; such trading would bring the prices back into line. Based on this simple principle, called *absence of arbitrage,* Black and Scholes (1973) derived the now famous formula for the value of the European call-option, which bears their name and which was extended by Merton (1973) in a variety of very significant ways. For this foundational work, Robert Merton and Myron Scholes were awarded the 1997 Nobel Prize in economics.

While options and other derivative securities can be used for speculation, their primary appeal is to investors who want to remove some of the risk associated with their investments or businesses. The sellers of derivative securities are faced with the twin problems of *pricing* and *hedging* them, and to accomplish this, current practice is to use Brownian-motion-based models of asset prices. Without such models and the analytical tractability that they provide, the market for derivative securities could not have grown to its present mammoth proportions.

Before proceeding further in this brief description of modern finance, there are two myths about the mathematical theory of finance that we need to explode.

The first myth is that this research is only about how to "beat the market." It is true that much of the portfolio optimization work growing out of the first revolution in finance is about how to "beat the market," but a substantial component is about how to understand the market for other purposes, such as regulation. The second revolution in finance, the derivative securities explosion, is not about beating the market at all.

The second myth maintains that since the finance industry does not manufacture tangible commodities, such as refrigerators or automobiles, it can be engaged in nothing but a zero-sum game, "robbing Peter to pay Paul." In fact, the role of financial institutions in a decentralized economy is to facilitate the flow of capital to sectors of society engaged in production. An efficient finance industry will facilitate this flow at the least possible cost, making available to the manufacturing sector a wide variety of instruments for borrowing and investing.

Consider, for example, a manufacturer who contemplates expansion of his production facilities and who chooses to finance this expansion by borrowing capital, in effect taking a mortgage on the new facilities. The terms (e.g., fixed or variable interest rate, term, prepayment options, collateralization) under which the manufacturer is willing to borrow money may not neatly match the terms under which any particular lender is willing to provide it. The finance industry should take the investments that lenders are willing to make, restructuring and recombining them as necessary, so as to provide a loan the manufacturer is willing to accept. The finance industry should perform this function in a wide variety of settings and manage its affairs so as to be exposed to minimal risk.

Let us suppose that the manufacturer is unable to plan effectively if he takes out a variable-rate mortgage, and so insists on a fixed-rate mortgage. Imagine also that an investment bank makes the mortgage, using money invested with it by depositors expecting to receive payments at the current (variable) interest rate. The bank is obliged to make monthly payments to these investors; the amounts of these payments fluctuate with the prevailing interest rates, and may be larger or smaller than what the bank receives from the manufacturer. To remove the risk associated with this position, the bank constructs a *hedge*. It may, for example, choose to sell short a number of bonds, i.e., receive money now in exchange for a promise to deliver bonds that it does not presently own and will have to buy eventually. If interest rates rise, the bank will have to pay its investors more than it receives on the loan from the manufacturer, but the cost of buying the bonds it has promised to deliver will fall. If the bank chooses its position carefully, its additional liability to its investors will be exactly offset by the downward movement of bond prices, and it will thus be protected against increases in the interest rate. Of course, decreases in interest rates will cause bond prices to rise, and the bank should choose its hedging position so as to be protected against this eventuality as well.

As one can see from this overly simplistic example, *a proliferation of financial instruments can enhance the efficiency of an economy.* The bank in this example "synthesizes" a fixed-rate mortgage using variable-rate investments and a position in the bond market. Such synthetic securities are the "products" of investment banks; while no one would claim that every "product" of this type contributes to the well-being of the nation, there is no doubt that an economy that has available a large variety of such products has a comparative advantage over one with a more limited offering. The firm that "manufactures" such products can do so only if it has reliable *models* for pricing and hedging them. Current models are built using stochastic calculus, are fit to the data by careful statistical estimation procedures, and require accurate and fast real-time numerical analysis.

This book is about some of these models. It treats only a small part of the whole picture, leaving completely untouched the issues of estimation and numerical analysis. Even within the range of models used in finance, we have found it necessary to be selective. Our guide has been to write about what we know best, namely areas of research in which we have had some level of personal involvement. Through the inclusion of an extensive bibliography and of notes at the end of each chapter, we have tried to point the reader toward some of the topics not touched. The bibliography is necessarily incomplete. We apologize to those whose work should have been included but is not. Such omissions are unintentional, and due either to ignorance or oversight.

In order to read this book one should be familiar with the material contained in the first three chapters of our book *Brownian Motion and Stochastic Calculus* (Springer-Verlag, New York, 1991). There are many other good sources for this material, but we will refer to the source we know best when we cite specific results.

Here is a high-level overview of the contents of this monograph. In *Chapter 1* we set up the generally accepted, Brownian-motion-driven model for financial markets. Because the coefficient processes in this model are themselves stochastic process, this is nearly the most general continuous-time model conceivable among those in which prices move continuously. The model of Chapter 1 allows us to introduce notions and results about portfolio and consumption rules, arbitrage, equivalent martingale measures, and attainability of contingent claims; it divides naturally into two cases, called *complete* and *incomplete*, respectively.

Chapter 2 lays out the theory of pricing and hedging contingent claims (the "synthetic" or "derivative" securities described above) in the context of a *complete market.* To honor the origins of the subject and to acquaint the reader with some important special cases, we analyze in some detail the pricing and hedging of a number of different options. We have also included a section on "futures" contracts, derivative securities that are conceptually more difficult because their value is defined recursively.

Chapter 3 takes up the problem of a single agent faced with optimal consumption and investment decisions in the complete version of the market model in Chapter 1. Tools from stochastic calculus and partial differential equations of parabolic type permit a very general treatment of the associated optimization problem. This theory can be related to Markowitz's mean–variance analysis and is ostensibly about how to "beat the market," although another important use for it is as a first step toward understanding how markets operate. Its latter use is predicated on the principle that a good model of individual behavior is to postulate that individuals act in their own best interest.

Chapter 4 carries the notions and results of Chapter 3 to their logical conclusion. In particular, it is assumed that there are several individuals in the economy, each behaving as described in Chapter 3; through the *law of supply and demand*, their collective actions determine the so-called *equilibrium* prices of securities in the market. Characterization of this equilibrium permits the study of questions about the effect of interventions in the market.

In *Chapter 5* we turn to the more difficult issue of pricing and hedging contingent claims in markets with *incompleteness* or other *constraints* on individual investors' portfolio choices. An approach based on "fictitious completion" for such a market, coupled with notions and results from *convex analysis* and *duality theory*, permits again a very general solution to the hedging problem.

Finally, *Chapter 6* uses the approach developed in Chapter 5 to treat the optimal consumption/investment problem for such incomplete or constrained markets, and for markets with different interest rates for borrowing and investing.

Note to the Reader

We use a hierarchical numbering system for equations and statements. The k-th equation in Section j of Chapter i is labeled $(j.k)$ at the place where it occurs and is cited as $(j.k)$ within Chapter i, but as $(i.j.k)$ outside Chapter i. A definition, theorem, lemma, corollary, remark, or exercise is a "statement," and the k-th statement in Section j of Chapter i is labeled $j.k$ *Statement* at the place where it occurs, and is cited as *Statement* $j.k$ within Chapter i but as *Statement* $i.j.k$ outside Chapter i.

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Ioannis Karatzas
Steven E. Shreve

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