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Brownian Motion and Stochastic Calculus

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To Eleni and Dot

Preface

Two of the most fundamental concepts in the theory of stochastic processes are the *Markov property* and the *martingale property*.* This book is written for readers who are acquainted with both of these ideas in the discrete-time setting, and who now wish to explore stochastic processes in their continuous-time context. It has been our goal to write a systematic and thorough exposition of this subject, leading in many instances to the frontiers of knowledge. At the same time, we have endeavored to keep the mathematical prerequisites as low as possible, namely, knowledge of measure-theoretic probability and some familiarity with discrete-time processes. The vehicle we have chosen for this task is *Brownian motion*, which we present as the canonical example of both a Markov process and a martingale. We support this point of view by showing how, by means of stochastic integration and random time change, all continuous-path martingales and a multitude of continuous-path Markov processes can be represented in terms of Brownian motion. This approach forces us to leave aside those processes which do not have continuous paths. Thus, the Poisson process is not a primary object of study, although it is developed in Chapter 1 to be used as a tool when we later study passage times and local time of Brownian motion.

The text is organized as follows: Chapter 1 presents the basic properties of martingales, as they are used throughout the book. In particular, we generalize from the discrete to the continuous-time context the martingale convergence theorem, the optional sampling theorem, and the Doob–Meyer decomposition. The latter gives conditions under which a submartingale can be written

* According to M. Loève, "martingales, Markov dependence and stationarity are the only three dependence concepts so far isolated which are sufficiently general and sufficiently amenable to investigation, yet with a great number of deep properties" (*Ann. Probab.* **1** (1973), p. 6).

as the sum of a martingale and an increasing process, and associates to every martingale with continuous paths a "quadratic variation process." This process is instrumental in the construction of stochastic integrals with respect to continuous martingales.

Chapter 2 contains three different constructions of Brownian motion, as well as discussions of the Markov and strong Markov properties for continuous-time processes. These properties are motivated by d-dimensional Brownian motion, but are developed in complete generality. This chapter also contains a careful discussion of the various filtrations commonly associated with Brownian motion. In Section 2.8 the strong Markov property is applied to a study of one-dimensional Brownian motion on a half-line, and on a bounded interval with absorption and reflection at the endpoints. Many densities involving first passage times, last exit times, absorbed Brownian motion, and reflected Brownian motion are explicitly computed. Section 2.9 is devoted to a study of sample path properties of Brownian motion. Results found in most texts on this subject are included, and in addition to these, a complete proof of the Lévy modulus of continuity is provided.

The theory of stochastic integration with respect to continuous martingales is developed in Chapter 3. We follow a middle path between the original constructions of stochastic integrals with respect to Brownian motion and the more recent theory of stochastic integration with respect to right-continuous martingales. By avoiding discontinuous martingales, we obviate the need to introduce the concept of predictability and the associated, highly technical, measure-theoretic machinery. On the other hand, it requires little extra effort to consider integrals with respect to continuous martingales rather than merely Brownian motion. The remainder of Chapter 3 is a testimony to the power of this more general approach; in particular, it leads to strong theorems concerning representations of continuous martingales in terms of Brownian motion (Section 3.4). In Section 3.3 we develop the chain rule for stochastic calculus, commonly known as Itô's formula. The Girsanov Theorem of Section 3.5 provides a method of changing probability measures so as to alter the drift of a stochastic process. It has become an indispensable method for constructing solutions of stochastic differential equations (Section 5.3) and is also very important in stochastic control (e.g., Section 5.8) and filtering. Local time is introduced in Sections 3.6 and 3.7, and it is shown how this concept leads to a generalization of the Itô formula to convex but not necessarily differentiable functions.

Chapter 4 is a digression on the connections between Brownian motion, Laplace's equation, and the heat equation. Sharp existence and uniqueness theorems for both these equations are provided by probabilistic methods; applications to the computation of boundary crossing probabilities are discussed, and the formulas of Feynman and Kac are established.

Chapter 5 returns to our main theme of stochastic integration and differential equations. In this chapter, stochastic differential equations are driven by Brownian motion and the notions of *strong* and *weak* solutions are presented. The basic Itô theory for strong solutions and some of its ramifications, including comparison and approximation results, are offered in Section 5.2, whereas Section 5.3 studies weak solutions in the spirit of Yamada & Watanabe. Essentially equivalent to the search for a weak solution is the search for a solution to the "Martingale Problem" of Stroock & Varadhan. In the context of this martingale problem, a full discussion of existence, uniqueness, and the strong Markov property for solutions of stochastic differential equations is given in Section 5.4. For one-dimensional equations it is possible to provide a complete characterization of solutions which exist only up to an "explosion time," and this is set forth in Section 5.5. This section also presents the recent and quite striking results of Engelbert & Schmidt concerning existence and uniqueness of solutions to one-dimensional equations. This theory makes substantial use of the local time material of Sections 3.6, 3.7 and the martingale representation results of Subsections 3.4.A,B. By analogy with Chapter 4, we discuss in Section 5.7 the connections between solutions to stochastic differential equations and elliptic and parabolic partial differential equations. Applications of many of the ideas in Chapters 3 and 5 are contained in Section 5.8, where we discuss questions of option pricing and optimal portfolio/consumption management. In particular, the Girsanov theorem is used to remove the difference between average rates of return of different stocks, a martingale representation result provides the optimal portfolio process, and stochastic representations of solutions to partial differential equations allow us to recast the optimal portfolio and consumption management problem in terms of two linear parabolic partial differential equations, for which explicit solutions are provided.

Chapter 6 is for the most part derived from Paul Lévy's profound study of Brownian excursions. Lévy's intuitive work has now been formalized by such notions as filtrations, stopping times, and Poisson random measures, but the remarkable fact remains that he was able, 40 years ago and working without these tools, to penetrate into the *fine structure of the Brownian path* and to inspire all the subsequent research on these matters until today. In the spirit of Lévy's work, we show in Section 6.2 that when one travels along the Brownian path with a clock run by the local time, the number of excursions away from the origin that one encounters, whose duration exceeds a specified number, has a Poisson distribution. Lévy's heuristic construction of Brownian motion from its excursions has been made rigorous by other authors. We do not attempt such a construction here, nor do we give a complete specification of the distribution of Brownian excursions; in the interest of intelligibility, we content ourselves with the specification of the distribution for the durations of the excursions. Sections 6.3 and 6.4 derive distributions for functionals of Brownian motion involving its local time; we present, in particular, a Feynman–Kac result for the so-called "elastic" Brownian motion, the formulas of D. Williams and H. Taylor, and the Ray–Knight description of

Brownian local time. An application of this theory is given in Section 6.5, where a one-dimensional stochastic control problem of the "bang-bang" type is solved.

The writing of this book has become for us a monumental undertaking involving several people, whose assistance we gratefully acknowledge. Foremost among these are the members of our families, Eleni, Dot, Andrea, and Matthew, whose support, encouragement, and patience made the whole endeavor possible. Parts of the book grew out of notes on lectures given at Columbia University over several years, and we owe much to the audiences in those courses. The inclusion of several exercises, the approaches taken to a number of theorems, and several citations of relevant literature resulted from discussions and correspondence with F. Baldursson, A. Dvoretzky, W. Fleming, O. Kallenberg, T. Kurtz, S. Lalley, J. Lehoczky, D. Stroock, and M. Yor. We have also taken exercises from Mandl, Lánská & Vrkoč (1978), and Ethier & Kurtz (1986). As the project proceeded, G.-L. Xu, Z.-L. Ying, and Th. Zariphopoulou read large portions of the manuscript and suggested numerous corrections and improvements. Careful reading by Daniel Ocone and Manfred Schäl revealed minor errors in the first printing, and these have been corrected. Others, including F. Åkesson, S. Dayanik, B. Doytchinov, H.J. Engelbert, R. Höhnle, C. Hou, A. Karolik, W. Nichols, L. Nielsen, D. Ocone, N. Vaillant and H. Wang found errors and/or contributed ideas, which have resulted in improvements in subsequent printings. However, our greatest single debt of gratitude goes to Marc Yor, who read much of the near-final draft and offered substantial mathematical and editorial comments on it. The typing was done tirelessly, cheerfully, and efficiently by Stella DeVito and Doodmatie Kalicharan; they have our most sincere appreciation.

We are grateful to Sanjoy Mitter and Dimitri Bertsekas for extending to us the invitation to spend the critical initial year of this project at the Massachusetts Institute of Technology. During that time the first four chapters were essentially completed, and we were partially supported by the Army Research Office under grant DAAG-299-84-K-0005. Additional financial support was provided by the National Science Foundation under grants DMS-84-16736 and DMS-84-03166 and by the Air Force Office of Scientific Research under grants AFOSR 82-0259, AFOSR 85-0360, and AFOSR 86-0203.

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Contents

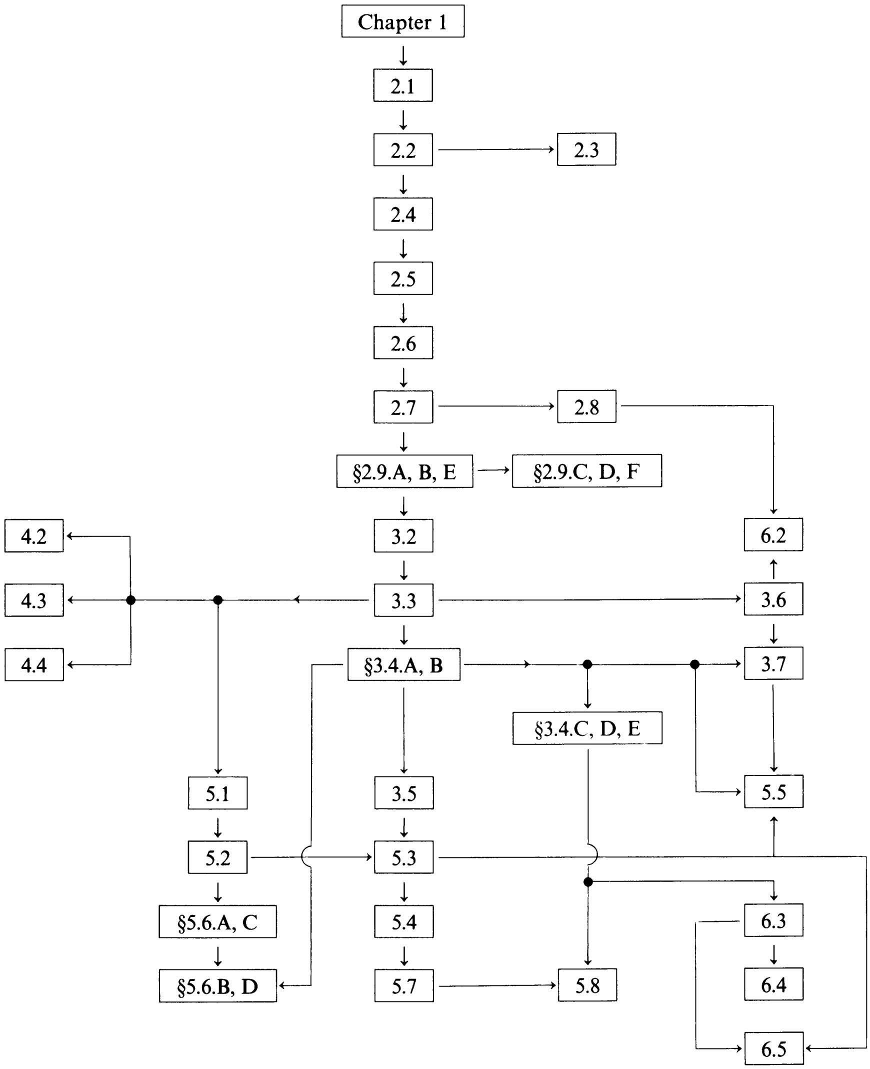
Suggestions for the Reader

We use a hierarchical numbering system for equations and statements. The k-th equation in Section j of Chapter i is labeled $(j.k)$ at the place where it occurs and is cited as $(j.k)$ within Chapter i, but as $(i.j.k)$ outside Chapter i. A definition, theorem, lemma, corollary, remark, problem, exercise, or solution is a "statement," and the k-th statement in Section j of Chapter i is labeled *j.k Statement* at the place where it occurs, and is cited as *Statement j.k* within Chapter i but as *Statement i.j.k* outside Chapter i.

This book is intended as a text and can be used in either a one-semester or a two-semester course, or as a text for a special topic seminar. The accompanying figure shows dependences among sections, and in some cases among subsections. In a one-semester course, we recommend inclusion of Chapter 1 and Sections 2.1, 2.2, 2.4, 2.5, 2.6, 2.7, §2.9.A, B, E, Sections 3.2, 3.3, 5.1, 5.2, and §5.6.A, C. This material provides the basic theory of stochastic integration, including the Itô calculus and the basic existence and uniqueness results for strong solutions of stochastic differential equations. It also contains matters of interest in engineering applications, namely, Fisk–Stratonovich integrals and approximation of stochastic differential equations in §3.3.A and 5.2.D, and Gauss–Markov processes in §5.6.A. Progress through this material can be accelerated by omitting the proof of the Doob–Meyer Decomposition Theorem 1.4.10 and the proofs in §2.4.D. The statements of Theorem 1.4.10, Theorem 2.4.20, Definition 2.4.21, and Remark 2.4.22 should, however, be retained. If possible in a one-semester course, and certainly in a two-semester course, one should include the topic of weak solutions of stochastic differential equations. This is accomplished by covering §3.4.A, B, and Sections 3.5, 5.3, and 5.4. Section 5.8 serves as an introduction to *stochastic control*, and so we recommend adding §3.4.C, D, E, and Sections 5.7, and 5.8 if time permits. In either a one- or two-semester course, Section 2.8 and part or all of Chapter 4 may be included according to time and interest. The material on *local time* and its applications in Sections 3.6, 3.7, 5.5, and in Chapter 6 would normally be the subject of a special topic course with advanced students.

The text contains about 175 "problems" and over 100 "exercises." The former are assignments to the reader to fill in details or generalize a result, and these are often quoted later in the text. We judge approximately two-thirds of these problems to be nontrivial or of fundamental importance, and solutions for such problems are provided at the end of each chapter. The exercises are also often significant extensions of results developed in the text, but these will not be needed later, except perhaps in the solution of other exercises. Solutions for the exercises are not provided. There are some exercises for which the solution we know violates the dependencies among sections shown in the figure, but such violations are pointed out in the offending exercises, usually in the form of a hint citing an earlier result.

Interdependence of the Chapters



Frequently Used Notation

I. General Notation

Let a and b be real numbers.
(1) $\triangleq$ means "is defined to be."
(2) $a \wedge b \triangleq \min\{a, b\}$.
(3) $a \vee b \triangleq \max\{a, b\}$.
(4) $a^+ \triangleq \max\{a, 0\}$.
(5) $a^- \triangleq \max\{-a, 0\}$.

II. Sets and Spaces

(1) $\mathbb{N}_0 \triangleq \{0, 1, 2, \ldots\}$.
(2) Q is the set of rational numbers.
(3) Q^+ is the set of nonnegative rational numbers.
(4) $\mathbb{R}^d$ is the d-dimensional Euclidean space; $\mathbb{R}^1 = \mathbb{R}$.
(5) $B_r \triangleq \{x \in \mathbb{R}^d;\ \|x\| < r\}$ (p. 240).
(6) $(\mathbb{R}^d)^{[0,\infty)}$ is the set of functions from $[0, \infty)$ to $\mathbb{R}^d$ (pp. 49, 76).
(7) $C[0, \infty)^d$ is the subspace of $(\mathbb{R}^d)^{[0,\infty)}$ consisting of continuous functions; $C[0, \infty)^1 = C[0, \infty)$ (pp. 60, 64).
(8) $D[0, \infty)$ is the subspace of $\mathbb{R}^{[0,\infty)}$ consisting of functions which are right continuous and have left-limits (p. 409).
(9) $C^k(E)$, $C_b^k(E)$, $C_0^k(E)$: See Remark 4.1, p. 312.
(10) $C^{1,2}([0, T) \times E)$, $C^{1,2}((0, T) \times E)$: See Remark 4.1, p. 312.
(11) $\mathscr{L}$, $\mathscr{L}(M)$, $\mathscr{L}^*$, $\mathscr{L}^*(M)$: See pp. 130–131.
(12) $\mathscr{P}$, $\mathscr{P}(M)$, $\mathscr{P}^*$, $\mathscr{P}^*(M)$: See pp. 146–147.
(13) $\mathscr{M}_2(\mathscr{M}_2^c)$: The space of (continuous) square-integrable martingales (p. 30).
(14) $\mathscr{M}^{\mathrm{loc}}(\mathscr{M}^{c,\mathrm{loc}})$: The space of (continuous) local martingales (p. 36).

III. Functions

(1) $\operatorname{sgn}(x) = \begin{cases} 1; & x > 0, \\ -1; & x \le 0. \end{cases}$

(2) $1_A(x) \triangleq \begin{cases} 1; & x \in A, \\ 0; & x \notin A. \end{cases}$

(3) $p(t; x, y) \triangleq \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$; $t > 0$, $x, y \in \mathbb{R}$ (p. 52).

(4) $p_{\pm}(t; x, y) \triangleq p(t; x, y) \pm p(t; x, -y)$; $t > 0$, $x, y \in \mathbb{R}$ (p. 97).

(5) $[\![t]\!]$ is the largest integer less than or equal to the real number t.

IV. σ-Fields

(1) $\mathscr{B}(U)$: The smallest σ-field containing all open sets of the topological space U (p. 1).

(2) $\mathscr{B}_t(C[0, \infty))$, $\mathscr{B}_t(C[0, \infty)^d)$. See pp. 60, 307.

(3) $\sigma(\mathscr{G})$: The smallest σ-field containing the collection of sets $\mathscr{G}$.

(4) $\sigma(X_s)$: The smallest σ-field with respect to which the random variable X_s is measurable.

(5) $\sigma(X_s; 0 \le s \le t)$: The smallest σ-field with respect to which the random variable X_s is measurable, $\forall s \in [0, t]$.

(6) $\mathscr{F}_t^X \triangleq \sigma(X_s; 0 \le s \le t)$, $\mathscr{F}_\infty \triangleq \sigma(\bigcup_{t \ge 0} \mathscr{F}_t)$: See p. 3.

(7) $\mathscr{F}_{t+} \triangleq \bigcap_{\varepsilon > 0} \mathscr{F}_{t+\varepsilon}$, $\mathscr{F}_{t-} \triangleq \sigma(\bigcup_{s<t} \mathscr{F}_s)$: See p. 4.

(8) $\mathscr{F}_T$: The σ-field of events determined prior to the stopping time T; see p. 8.

(9) $\mathscr{F}_{T+}$: The σ-field of events determined immediately after the optional time T; see p. 10.

(10) $\mathscr{F} \otimes \mathscr{G} \triangleq \sigma(A \times B; A \in \mathscr{F}, B \in \mathscr{G}\}$: The product σ-field formed from the σ-fields $\mathscr{F}$ and $\mathscr{G}$.

V. Operations on Functions

(1) $\Delta \triangleq \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$: The Laplacian (p. 240).

(2) $\mathscr{A}$, $\mathscr{A}_t$: Second order differential operators; see pp. 281, 311.

VI. Operations on Processes

(1) θ_s, θ_S: Shift operator at the deterministic time s and the random time S; see pp. 77, 83.

(2) $I_t^M(X) \triangleq \int_0^t X_s \, dM_s$: The stochastic integral of X with respect to M. See p. 141 for $M \in \mathscr{M}_2^c$, $X \in \mathscr{L}^*(M)$; see p. 147 for $M \in \mathscr{M}^{c,\text{loc}}$, $X \in \mathscr{P}^*(M)$.
(3) $M_t^* \triangleq \max_{0 \le s \le t} |M_s|$: See p. 163 for $M \in \mathscr{M}^{c,\text{loc}}$.
(4) $\langle X \rangle$: The quadratic variation process of $X \in \mathscr{M}_2$ (p. 31) or $X \in \mathscr{M}^{c,\text{loc}}$ (p. 36).
(5) $\langle X, Y \rangle$: The cross-variation process of X, Y in $\mathscr{M}_2$ (p. 31) or in $\mathscr{M}^{c,\text{loc}}$ (p. 36).
(6) $\|X\|_t$, $\|X\|$: See p. 37 for $X \in \mathscr{M}_2$.

VII. Miscellaneous

(1) $m_T(X, \delta) \triangleq \sup\{|X_s - X_t|; 0 \le s < t \le T, t - s \le \delta\}$; See p. 33.
(2) $m^T(\omega, \delta) \triangleq \max\{|\omega(s) - \omega(t)|; 0 \le s < t \le T, t - s \le \delta\}$: See p. 62.
(3) $\bar{D}$: The closure of the set $D \subset \mathbb{R}^d$.
(4) D^c: The complement of the set D.
(5) ∂D: The boundary of the set $D \subset \mathbb{R}^d$.
(6) $\tau_D \triangleq \inf\{t \ge 0; W_t \in D^c\}$: The first time the Brownian motion W exits from the set $D \subset \mathbb{R}^d$ (p. 240).
(7) $T_b \triangleq \inf\{t \ge 0; W_t = b\}$: The first time the one-dimensional Brownian motion W reaches the level $b \in \mathbb{R}$ (p. 79).
(8) $\Gamma_+(t) \triangleq \int_0^t 1_{(0,\infty)}(W_s)\, ds$: The occupation time by Brownian motion of the positive half-line (p. 273).
(9) $P_n \xrightarrow{w} P$: Weak convergence of the sequence of probability measures $\{P_n\}_{n=1}^\infty$ to the probability measure P (p. 60).
(10) $X_n \xrightarrow{\mathscr{D}} X$: Convergence in distribution of the sequence of random variables $\{X_n\}_{n=1}^\infty$ to the random variable X (p. 61).
(11) P^x: Probability measure corresponding to Brownian motion (p. 72) or a Markov process (p. 74) with initial position $x \in \mathbb{R}^d$.
(12) P^μ: Probability measure corresponding to Brownian motion (p. 72) or a Markov process (p. 74) with initial distribution μ.
(13) $\mathscr{N}_t^\mu$, $\mathscr{N}^\mu$: Collections of P^μ-negligible sets (p. 89).
(14) $I(\sigma)$, $Z(\sigma)$: See pp. 331, 332.
(15) I_d: The $(d \times d)$ identity matrix.
(16) meas: Lebesgue measure on the real line (p. 105).