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Olav Kallenberg

Foundations of Modern Probability



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Preface

Some thirty years ago it was still possible, as Loève so ably demonstrated, to write a single book in probability theory containing practically everything worth knowing in the subject. The subsequent development has been explosive, and today a corresponding comprehensive coverage would require a whole library. Researchers and graduate students alike seem compelled to a rather extreme degree of specialization. As a result, the subject is threatened by disintegration into dozens or hundreds of subfields.

At the same time the interaction between the areas is livelier than ever, and there is a steadily growing core of key results and techniques that every probabilist needs to know, if only to read the literature in his or her own field. Thus, it seems essential that we all have at least a general overview of the whole area, and we should do what we can to keep the subject together. The present volume is an earnest attempt in that direction.

My original aim was to write a book about “everything.” Various space and time constraints forced me to accept more modest and realistic goals for the project. Thus, “foundations” had to be understood in the narrower sense of the early 1970s, and there was no room for some of the more recent developments. I especially regret the omission of topics such as large deviations, Gibbs and Palm measures, interacting particle systems, stochastic differential geometry, Malliavin calculus, SPDEs, measure-valued diffusions, and branching and superprocesses. Clearly plenty of fundamental and intriguing material remains for a possible second volume.

Even with my more limited, revised ambitions, I had to be extremely selective in the choice of material. More importantly, it was necessary to look for the most economical approach to every result I did decide to include. In the latter respect, I was surprised to see how much could actually be done to simplify and streamline proofs, often handed down through generations of textbook writers. My general preference has been for results conveying some new idea or relationship, whereas many propositions of a more technical nature have been omitted. In the same vein, I have avoided technical or computational proofs that give little insight into the proven results. This conforms with my conviction that the logical structure is what matters most in mathematics, even when applications is the ultimate goal.

Though the book is primarily intended as a general reference, it should also be useful for graduate and seminar courses on different levels, ranging from elementary to advanced. Thus, a first-year graduate course in measure-theoretic probability could be based on the first ten or so chapters, while the rest of the book will readily provide material for more advanced courses on various topics. Though the treatment is formally self-contained, as far as measure theory and probability are concerned, the text is intended for a rather sophisticated reader with at least some rudimentary knowledge of subjects like topology, functional analysis, and complex variables.

My exposition is based on experiences from the numerous graduate and seminar courses I have been privileged to teach in Sweden and in the United States, ever since I was a graduate student myself. Over the years I have developed a personal approach to almost every topic, and even experts might find something of interest. Thus, many proofs may be new, and every chapter contains results that are not available in the standard textbook literature. It is my sincere hope that the book will convey some of the excitement I still feel for the subject, which is without a doubt (even apart from its utter usefulness) one of the richest and most beautiful areas of modern mathematics.

Notes and Acknowledgments: My first thanks are due to my numerous Swedish teachers, and especially to Peter Jagers, whose 1971 seminar opened my eyes to modern probability. The idea of this book was raised a few years later when the analysts at Gothenburg asked me to give a short lecture course on “probability for mathematicians.” Although I objected to the title, the lectures were promptly delivered, and I became convinced of the project’s feasibility. For many years afterward I had a faithful and enthusiastic audience in numerous courses on stochastic calculus, SDEs, and Markov processes. I am grateful for that learning opportunity and for the feedback and encouragement I received from colleagues and graduate students.

Inevitably I have benefited immensely from the heritage of countless authors, many of whom are not even listed in the bibliography. I have further been fortunate to know many prominent probabilists of our time, who have often inspired me through their scholarship and personal example. Two people, Klaus Matthes and Gopi Kallianpur, stand out as particularly important influences in connection with my numerous visits to Berlin and Chapel Hill, respectively.

The great Kai Lai Chung, my mentor and friend from recent years, offered penetrating comments on all aspects of the work: linguistic, historical, and mathematical. My colleague Ming Liao, always a stimulating partner for discussions, was kind enough to check my material on potential theory. Early versions of the manuscript were tested on several groups of graduate students, and Kamesh Casukhela, Davorin Dujmovic, and Hussain Talibi in particular were helpful in spotting misprints. Ulrich Albrecht and Ed Slaminka offered generous help with software problems. I am further grateful to John Kimmel, Karina Mikhli, and the Springer production team for their patience with my last-minute revisions and their truly professional handling of the project.

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Olav Kallenberg
May 1997

Contents

1. Elements of Measure Theory	1
<i>σ-fields and monotone classes</i>	
<i>measurable functions</i>	
<i>measures and integration</i>	
<i>monotone and dominated convergence</i>	
<i>transformation of integrals</i>	
<i>product measures and Fubini's theorem</i>	
<i>L^p-spaces and projection</i>	
<i>measure spaces and kernels</i>	
2. Processes, Distributions, and Independence	22
<i>random elements and processes</i>	
<i>distributions and expectation</i>	
<i>independence</i>	
<i>zero-one laws</i>	
<i>Borel–Cantelli lemma</i>	
<i>Bernoulli sequences and existence</i>	
<i>moments and continuity of paths</i>	
3. Random Sequences, Series, and Averages	39
<i>convergence in probability and in L^p</i>	
<i>uniform integrability and tightness</i>	
<i>convergence in distribution</i>	
<i>convergence of random series</i>	
<i>strong laws of large numbers</i>	
<i>Portmanteau theorem</i>	
<i>continuous mapping and approximation</i>	
<i>coupling and measurability</i>	
4. Characteristic Functions and Classical Limit Theorems	60
<i>uniqueness and continuity theorem</i>	
<i>Poisson convergence</i>	
<i>positive and symmetric terms</i>	
<i>Lindeberg's condition</i>	
<i>general Gaussian convergence</i>	
<i>weak laws of large numbers</i>	
<i>domain of Gaussian attraction</i>	
<i>vague and weak compactness</i>	
5. Conditioning and Disintegration	80
<i>conditional expectations and probabilities</i>	
<i>regular conditional distributions</i>	

disintegration theorem
conditional independence
transfer and coupling
Daniell–Kolmogorov theorem
extension by conditioning

6. Martingales and Optional Times 96

filtrations and optional times
random time-change
martingale property
optional stopping and sampling
maximum and upcrossing inequalities
martingale convergence, regularity, and closure
limits of conditional expectations
regularization of submartingales

7. Markov Processes and Discrete-Time Chains 117

Markov property and transition kernels
finite-dimensional distributions and existence
space homogeneity and independence of increments
strong Markov property and excursions
invariant distributions and stationarity
recurrence and transience
ergodic behavior of irreducible chains
mean recurrence times

8. Random Walks and Renewal Theory 136

recurrence and transience
dependence on dimension
general recurrence criteria
symmetry and duality
Wiener–Hopf factorization
ladder time and height distribution
stationary renewal process
renewal theorem

9. Stationary Processes and Ergodic Theory 156

stationarity, invariance, and ergodicity
mean and a.s. ergodic theorem
continuous time and higher dimensions
ergodic decomposition
subadditive ergodic theorem
products of random matrices
exchangeable sequences and processes
predictable sampling

10. Poisson and Pure Jump-Type Markov Processes	176
<i>existence and characterizations of Poisson processes</i>	
<i>Cox processes, randomization and thinning</i>	
<i>one-dimensional uniqueness criteria</i>	
<i>Markov transition and rate kernels</i>	
<i>embedded Markov chains and explosion</i>	
<i>compound and pseudo-Poisson processes</i>	
<i>Kolmogorov's backward equation</i>	
<i>ergodic behavior of irreducible chains</i>	
11. Gaussian Processes and Brownian Motion	199
<i>symmetries of Gaussian distribution</i>	
<i>existence and path properties of Brownian motion</i>	
<i>strong Markov and reflection properties</i>	
<i>arcsine and uniform laws</i>	
<i>law of the iterated logarithm</i>	
<i>Wiener integrals and isonormal Gaussian processes</i>	
<i>multiple Wiener–Itô integrals</i>	
<i>chaos expansion of Brownian functionals</i>	
12. Skorohod Embedding and Invariance Principles	220
<i>embedding of random variables</i>	
<i>approximation of random walks</i>	
<i>functional central limit theorem</i>	
<i>law of the iterated logarithm</i>	
<i>arcsine laws</i>	
<i>approximation of renewal processes</i>	
<i>empirical distribution functions</i>	
<i>embedding and approximation of martingales</i>	
13. Independent Increments and Infinite Divisibility	234
<i>regularity and jump structure</i>	
<i>Lévy representation</i>	
<i>independent increments and infinite divisibility</i>	
<i>stable processes</i>	
<i>characteristics and convergence criteria</i>	
<i>approximation of Lévy processes and random walks</i>	
<i>limit theorems for null arrays</i>	
<i>convergence of extremes</i>	
14. Convergence of Random Processes, Measures, and Sets	255
<i>relative compactness and tightness</i>	
<i>uniform topology on $C(K, S)$</i>	
<i>Skorohod's J_1-topology</i>	

equicontinuity and tightness
convergence of random measures
superposition and thinning
exchangeable sequences and processes
simple point processes and random closed sets

15. Stochastic Integrals and Quadratic Variation 275

continuous local martingales and semimartingales
quadratic variation and covariation
existence and basic properties of the integral
integration by parts and Itô's formula
Fisk–Stratonovich integral
approximation and uniqueness
random time-change
dependence on parameter

16. Continuous Martingales and Brownian Motion 296

martingale characterization of Brownian motion
random time-change of martingales
isotropic local martingales
integral representations of martingales
iterated and multiple integrals
change of measure and Girsanov's theorem
Cameron–Martin theorem
Wald's identity and Novikov's condition

17. Feller Processes and Semigroups 313

semigroups, resolvents, and generators
closure and core
Hille–Yosida theorem
existence and regularization
strong Markov property
characteristic operator
diffusions and elliptic operators
convergence and approximation

18. Stochastic Differential Equations and Martingale Problems 335

linear equations and Ornstein–Uhlenbeck processes
strong existence, uniqueness, and nonexplosion criteria
weak solutions and local martingale problems
well-posedness and measurability
pathwise uniqueness and functional solution
weak existence and continuity

transformations of SDEs
strong Markov and Feller properties

19. Local Time, Excursions, and Additive Functionals 350

Tanaka's formula and semimartingale local time
occupation density, continuity and approximation
regenerative sets and processes
excursion local time and Poisson process
Ray–Knight theorem
excessive functions and additive functionals
local time at regular point
additive functionals of Brownian motion

20. One-Dimensional SDEs and Diffusions 371

weak existence and uniqueness
pathwise uniqueness and comparison
scale function and speed measure
time-change representation
boundary classification
entrance boundaries and Feller properties
ratio ergodic theorem
recurrence and ergodicity

21. PDE-Connections and Potential Theory 390

backward equation and Feynman–Kac formula
uniqueness for SDEs from existence for PDEs
harmonic functions and Dirichlet's problem
Green functions as occupation densities
sweeping and equilibrium problems
dependence on conductor and domain
time reversal
capacities and random sets

22. Predictability, Compensation, and Excessive Functions 409

accessible and predictable times
natural and predictable processes
Doob–Meyer decomposition
quasi-left-continuity
compensation of random measures
excessive and superharmonic functions
additive functionals as compensators
Riesz decomposition

23. Semimartingales and General Stochastic Integration	433
<i>predictable covariation and L^2-integral</i>	
<i>semimartingale integral and covariation</i>	
<i>general substitution rule</i>	
<i>Doléans' exponential and change of measure</i>	
<i>norm and exponential inequalities</i>	
<i>martingale integral</i>	
<i>decomposition of semimartingales</i>	
<i>quasi-martingales and stochastic integrators</i>	
Appendices	455
A1. <i>Hard Results in Measure Theory</i>	
A2. <i>Some Special Spaces</i>	
Historical and Bibliographical Notes	464
Bibliography	486
Indices	509
<i>Authors</i>	
<i>Terms and Topics</i>	
<i>Symbols</i>	