## Calculus of Variations

Jürgen Jost and Xianqing Li-Jost

Max-Planck-Institute for Mathematics in the Sciences,

Leipzig



## Contents

ce ana summary	page x
arks on notation	xv
Part one: One-dimensional variational problems	1
The classical theory	3
The Euler-Lagrange equations. Examples	3
The idea of the direct methods and some regularity	
results	10
The second variation. Jacobi fields	18
Free boundary conditions	24
Symmetries and the theorem of E. Noether	26
A geometric example: geodesic curves	32
The length and energy of curves	32
Fields of geodesic curves	43
The existence of geodesics	51
Saddle point constructions	62
A finite dimensional example	62
The construction of Lyusternik–Schnirelman	67
The theory of Hamilton and Jacobi	79
The canonical equations	79
The Hamilton–Jacobi equation	81
Geodesics	87
Fields of extremals	89
Hilbert's invariant integral and Jacobi's theorem	92
Canonical transformations	95
	Part one: One-dimensional variational problems The classical theory The Euler-Lagrange equations. Examples The idea of the direct methods and some regularity results The second variation. Jacobi fields Free boundary conditions Symmetries and the theorem of E. Noether  A geometric example: geodesic curves The length and energy of curves Fields of geodesic curves The existence of geodesics  Saddle point constructions A finite dimensional example The construction of Lyusternik-Schnirelman The theory of Hamilton and Jacobi The canonical equations The Hamilton-Jacobi equation Geodesics Fields of extremals Hilbert's invariant integral and Jacobi's theorem

viii	Contents
VIII	Contents

<b>5</b>	Dynamic optimization	104
5.1	Discrete control problems	104
5.2	Continuous control problems	106
5.3	The Pontryagin maximum principle	109
	Part two: Multiple integrals in the calculus of	
	variations	115
1	Lebesgue measure and integration theory	117
1.1	The Lebesgue measure and the Lebesgue integral	117
1.2	Convergence theorems	122
2	Banach spaces	125
2.1	Definition and basic properties of Banach and Hilbert	
	spaces	125
2.2	Dual spaces and weak convergence	132
2.3	Linear operators between Banach spaces	144
2.4	Calculus in Banach spaces	150
3	$L^p$ and Sobolev spaces	159
3.1	$L^p$ spaces	159
3.2	Approximation of $L^p$ functions by smooth functions	
	(mollification)	166
3.3	Sobolev spaces	171
3.4	Rellich's theorem and the Poincaré and Sobolev	
	inequalities	175
4	The direct methods in the calculus of variations	183
4.1	Description of the problem and its solution	183
4.2	Lower semicontinuity	184
4.3	The existence of minimizers for convex variational	
	problems	187
4.4	Convex functionals on Hilbert spaces and Moreau-	
	Yosida approximation	190
4.5	The Euler–Lagrange equations and regularity questions	195
5	Nonconvex functionals. Relaxation	205
5.1	Nonlower semicontinuous functionals and relaxation	205
5.2	Representation of relaxed functionals via convex	
	envelopes	213
6	Γ-convergence	225
6.1	The definition of $\Gamma$ -convergence	225

	Contents	ix
6.2	Homogenization	231
6.3	Thin insulating layers	235
7	BV-functionals and $\Gamma$ -convergence: the example of	
	Modica and Mortola	<b>24</b> 1
7.1	The space $BV(\Omega)$	241
7.2	The example of Modica–Mortola	248
Apper	ndix A The coarea formula	257
Apper	ndix B The distance function from smooth hypersurfaces	262
8	Bifurcation theory	266
8.1	Bifurcation problems in the calculus of variations	266
8.2	The functional analytic approach to bifurcation theory	270
8.3	The existence of catenoids as an example of a bifurca-	
	tion process	282
9	The Palais-Smale condition and unstable critical	
	points of variational problems	<b>29</b> 1
9.1	The Palais-Smale condition	291
9.2	The mountain pass theorem	301
9.3	Topological indices and critical points	306

319

Index