Universitext

Springer-Verlag Berlin Heidelberg GmbH

Jürgen Jost

Riemannian Geometry and Geometric Analysis

Second Edition

Springer

Jürgen Jost
Max Planck Institute for Mathematics in the Sciences
Inselstr. 22-26
D-04103 Leipzig
Germany

Mathematics Subject Classification (1991):
53B21, 53C20, 32C17, 35I60, 49-XX, 58E20, 57R15

Library of Congress Cataloging-in Publication Data

Jost, Jürgen, 1956-
Riemannian geometry and geometric analysis / Jürgen Jost.2nd ed.
p. cm. -- (Universitext)
Includes bibliographical references and index.
ISBN 978-3-540-63654-0 ISBN 978-3-662-22385-7 (eBook)
DOI 10.1007/978-3-662-22385-7
1. Geometry, Riemannian. I. Title.
QA649.J67 1998
516.3'73--dc21 97-45047
CIP

ISBN 978-3-540-63654-0

Originally published by Springer-Verlag Berlin Heidelberg New York in 1998

Typesetting: By the author using Springer TEX macro package
Cover design: *design & production* GmbH, Heidelberg

SPIN: 10653871 41/3143 - 5 4 3 2 1 0 - Printed on acid-free paper

Dedicated to Shing-Tung Yau,
for so many discussions
about mathematics and Chinese culture

Preface to the Second Edition

In this new edition, I have added some new material, in particular on spin geometry, the Dirac operator, and the Seiberg-Witten equations. The Seiberg-Witten equations are satisfied by the absolute minima of a particular functional of the type occuring in quantum field theory under the appellation φ^4-theory, meaning that the nonlinearity is of fourth order in the field φ. The field is coupled to some kind of generalized electromagnetic potential A. The solution spaces of these equations have found very powerful mathematical applications. Thus, it seemed natural to include a brief description of these equations here, in particular a detailed motivation and derivation of them. As background material for that purpose, spin geometry and the Dirac operator are needed which of course are also topics of immense geometric interest by themselves.Several monographs are available that include a discussion of the algebra and geometry of spin structures (see the quotations in the corresponding sections of the present book), but in line with the character of the present textbook we have adopted here a more elementary and detailed discussion than can be found in research monographs. We thus hope to help the novice to get better aquainted with that material. Our treatment of the Seiberg-Witten equations is somewhat different from the existing literature, as we stress the analogy with the Ginzburg-Landau equations on a Riemann surface that are similar to the Seiberg-Witten equations in many respects, but more elementary and simpler to understand.

I have also taken the opportunity to correct a number of misprints and small inaccuracies in the first edition that were pointed out to me by several friends, and to make certain additions here and there.

Part of the new material is based on a course I taught at the University of Leipzig, and I thank the audience for their interest, and in particular Bernd Herzog and Guofang Wang for their stimulating questions and helpful comments. I am also grateful to Harald Wenk for his competent and efficient TEXwork.

Jürgen Jost

Preface

The present textbook is a somewhat expanded version of the material of a three-semester course I have given in Bochum. It attempts a synthesis of geometric and analytic methods in the study of Riemannian manifolds.

In the first chapter, we introduce the basic geometric concepts, like differentiable manifolds, tangent spaces, vector bundles, vector fields and one-parameter groups of diffeomorphisms, Lie algebras and groups and in particular Riemannian metrics. We also derive some elementary results about geodesics.

The second chapter introduces de Rham cohomology groups and the essential tools from elliptic PDE for treating these groups. In later chapters, we shall encounter nonlinear versions of the methods presented here.

The third chapter treats the general theory of connections and curvature.

In the fourth chapter, we introduce Jacobi fields, prove the Rauch comparison theorems for Jacobi fields and apply these results to geodesics.

These first four chapters treat the more elementary and basic aspects of the subject. Their results will be used in the remaining, more advanced chapters that are essentially independent of each other.

In the fifth chapter, we develop Morse theory and apply it to the study of geodesics.

The sixth chapter treats symmetric spaces as important examples of Riemannian manifolds in detail.

While these two chapters emphasize the geometric aspect, the next ones will be of a more analytical character. In the seventh chapter we take the study of geodesics up once more by developing the theory of critical points of the energy functional on the Sobolev space of loops of class $H^{1,2}$.

In the eighth chapter, we treat harmonic maps between Riemannian manifolds. We prove several existence theorems and apply them to Riemannian geometry.

A guiding principle for this textbook was that the material in the main body should be self contained. The essential exception is that we use material about Sobolev spaces and linear elliptic PDE without giving proofs. This material is collected in Appendix A. Appendix B collects some elementary topological results about fundamental groups and covering spaces. All the cohomology theory that is needed, however, is developed in the text

(Chapters 2 and 5). The only place where we deviate from our ground rule is in chapter 5 when we use the theorem from algebraic topology that any compact manifold has at least one nonvanishing homotopy group without proof. While that result is needed in order to show that any compact manifold admits a nontrivial closed geodesic, the idea of the proof can be readily apprehended without depending on that topological result.

We employ both coordinatefree intrinsic notations and tensor notations depending on local coordinates. We usually develop a concept in both notations while we sometimes alternate in the proofs. Besides not being a methodological purist, reasons for often prefering the tensor calculus to the more elegant and concise intrinsic one are the following. For the analytic aspects, one often has to employ results about (elliptic) partial differential equations (PDE), and in order to check that the relevant assumptions like ellipticity hold and in order to make contact with the notations usually employed in PDE theory, one usually has to write down the differential equation in local coordinates. Also, recently, manifold and important connections have been established between theoretical physics and our subject. In the physical literature, usually the tensor notation is prefered, and therefore, familiarity with that notation is necessary for exploring those connections that have been found to be stimulating for the development of mathematics, or promise to be so in the future.

As appendices to most of the paragraphs, we have written sections with the title "Perspectives". The aim of those sections is to place the material in a broader context and explain further results and directions without detailed proofs. The material of these Perspectives will not be used in the main body of the text. At the end of each chapter, some exercises for the reader are given. We trust the reader to be intelligent enough to understand our system of numbering and cross references without further explanation.

The development of the mathematical subject of Geometric Analysis, namely the investigation of analytical questions arising from a geometric context and in turn the application of analytical techniques to geometric problems, is to a large extent due to the work and the influence of Shing-Tung Yau. This book is dedicated to him.

I would like to thank Raimund Blache and Marianna Goldcheid for detailed corrections and many suggestions for improving the quality of the text. Some useful errata were also supplied by Wolfgang Medding, Markus Schlicht, Xiao-Wei Peng and Wilderich Tuschmann. I also thank Isolde Gottschlich for the excellent quality of the technical typing.

Jürgen Jost

Contents