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Arthur Jones Sidney A. Morris
Kenneth R. Pearson

Abstract Algebra and Famous Impossibilities

With 27 Illustrations

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Arthur Jones
Kenneth R. Pearson
Department of Mathematics
La Trobe University
Bundoora 3083
Australia

Sidney A. Morris
Faculty of Informatics
University of Wollongong
Wollongong 2500
Australia

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Preface

The famous problems of squaring the circle, doubling the cube and trisecting an angle captured the imagination of both professional and amateur mathematicians for over two thousand years. Despite the enormous effort and ingenious attempts by these men and women, the problems would not yield to purely geometrical methods. It was only the development of abstract algebra in the nineteenth century which enabled mathematicians to arrive at the surprising conclusion that these constructions are not possible.

In this book we develop enough abstract algebra to prove that these constructions are impossible. Our approach introduces all the relevant concepts about fields in a way which is more concrete than usual and which avoids the use of quotient structures (and even of the Euclidean algorithm for finding the greatest common divisor of two polynomials). Having the geometrical questions as a specific goal provides motivation for the introduction of the algebraic concepts and we have found that students respond very favourably.

We have used this text to teach second-year students at La Trobe University over a period of many years, each time refining the material in the light of student performance.

The text is pitched at a level suitable for students who have already taken a course in linear algebra, including the ideas of a vector space over a field, linear independence, basis and dimension. The treatment, in such a course, of fields and vector spaces as algebraic objects should provide an adequate background for the study of this book. Hence the book is suitable for Junior/Senior courses in North America and second-year courses in Australia.

Chapters 1 to 6, which develop the link between geometry and algebra, are the core of this book. These chapters contain a complete solution to the three famous problems, except for proving that π is a transcendental number (which is needed to complete the proof of the impossibility of squaring the circle). In Chapter 7 we give a self-contained proof that π is transcendental. Chapter 8 contains material about fields which is closely related to the topics in Chapters 2–4, although it is not required in the proof of the impossibility of the three constructions. The short concluding Chapter 9 describes some other areas of mathematics in which algebraic machinery can be used to prove impossibilities.

We expect that any course based on this book will include all of Chapters 1 – 6 and (ideally) at least passing reference to Chapter 9. We have often taught such a course which we cover in a term (about twenty hours). We find it essential for the course to be paced in a way that allows time for students to do a substantial number of problems for themselves. Different semester length (or longer) courses including topics from Chapters 7 and 8 are possible. The three natural parts of these are

(1) Sections 7.1 and 7.2 (transcendence of e),

(2) Sections 7.3 to 7.6 (transcendence of π),

(3) Chapter 8.

These are independent except, of course, that (2) depends on (1). Possible extensions to the basic course are to include one, two or all of these. While most treatments of the transcendence of π require familiarity with the theory of functions of a complex variable and complex integrals, ours in Chapter 7 is accessible to students who have completed the usual introductory real calculus course (first-year in Australia and Freshman/Sophomore in North America). However instructors should note that the arguments in Sections 7.3 to 7.6 are more difficult and demanding than those in the rest of the book.

Problems are given at the end of each section (rather than collected at the end of the chapter). Some of these are computational and others require students to give simple proofs.

Each chapter contains additional reading suitable for students and instructors. We hope that the text itself will encourage students to do further reading on some of the topics covered.

As in many books, exercises marked with an asterisk * are a good bit harder than the others. We believe it is important to identify clearly the end of each proof and we use the symbol ■ for this purpose.

We have found that students often lack the mathematical maturity required to write or understand simple proofs. It helps if students write down where the proof is heading, what they have to prove and how they might be able to prove it. Because this is not part of the formal proof, we indicate this exploration by separating it from the proof proper by using a box which looks like

(Include here what must be proved etc.)

Experience has shown that it helps students to use this material if important theorems are given specific names which suggest their content. We have enclosed these names in square brackets before the statement of the theorem. We encourage students to use these names when justifying their solutions to exercises. They often find it convenient to abbreviate the names to just the relevant initials. (For example, the name "Small Degree Irreducibility Theorem" can be abbreviated to S.D.I.T.)

We are especially grateful to our colleague Gary Davis, who pointed the way towards a more concrete treatment of field extensions (using residue rings rather than quotient rings) and thus made the course accessible to a wider class of students. We are grateful to Ernie Bowen, Jeff Brooks, Grant Cairns, Mike Canfell, Brian Davey, Alistair Gray, Paul Halmos, Peter Hodge, Alwyn Horadam, Deborah King, Margaret McIntyre, Bernhard H. Neumann, Kristen Pearson, Suzanne Pearson, Alf van der Poorten, Brailey Sims, Ed Smith and Peter Stacey, who have given us helpful feedback, made suggestions and assisted with the proof reading.

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A.J., S.A.M., K.R.P.
April 1991

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