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Lectures in Abstract Algebra

I. Basic Concepts

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**TO
MY WIFE**

PREFACE

The present volume is the first of three that will be published under the general title *Lectures in Abstract Algebra*. These volumes are based on lectures which the author has given during the past ten years at the University of North Carolina, at The Johns Hopkins University, and at Yale University. The general plan of the work is as follows: The present first volume gives an introduction to abstract algebra and gives an account of most of the important algebraic concepts. In a treatment of this type it is impossible to give a comprehensive account of the topics which are introduced. Nevertheless we have tried to go beyond the foundations and elementary properties of the algebraic systems. This has necessitated a certain amount of selection and omission. We feel that even at the present stage a deeper understanding of a few topics is to be preferred to a superficial understanding of many.

The second and third volumes of this work will be more specialized in nature and will attempt to give comprehensive accounts of the topics which they treat. Volume II will bear the title *Linear Algebra* and will deal with the theory of vector spaces. Volume III, *The Theory of Fields and Galois Theory*, will be concerned with the algebraic structure of fields and with valuations of fields.

All three volumes have been planned as texts for courses. A great many exercises of varying degrees of difficulty have been included. Some of these perhaps rate stars, but we have felt that the disadvantages of the system of starring difficult exercises outweigh its advantages. A few sections have been starred (notation: *1) to indicate that these can be omitted without jeopardizing the understanding of subsequent material.

We are indebted to a great many friends for helpful criticisms and encouragement during the course of preparation of this volume. Professors A. H. Clifford, G. Hochschild and R. E. Johnson, Drs. D. T. Finkbeiner and W. H. Mills have read parts of the manuscript and given us useful suggestions for improving it. Drs. Finkbeiner and Mills have assisted with the proofreading. I take this opportunity to offer my sincere thanks to all of these men.

N. J.

New Haven, Conn.
January 22, 1951

CONTENTS

INTRODUCTION: CONCEPTS FROM SET THEORY THE SYSTEM OF NATURAL NUMBERS

SECTION	PAGE
1. Operations on sets	2
2. Product sets, mappings	3
3. Equivalence relations	4
4. The natural numbers	7
5. The system of integers	10
6. The division process in I	12

CHAPTER I: SEMI-GROUPS AND GROUPS

1. Definition and examples of semi-groups	15
2. Non-associative binary compositions	18
3. Generalized associative law. Powers	20
4. Commutativity	21
5. Identities and inverses	22
6. Definition and examples of groups	23
7. Subgroups	24
8. Isomorphism	26
9. Transformation groups	27
10. Realization of a group as a transformation group	28
11. Cyclic groups. Order of an element	30
12. Elementary properties of permutations	34
13. Coset decompositions of a group	37
14. Invariant subgroups and factor groups	40
15. Homomorphism of groups	41
16. The fundamental theorem of homomorphism for groups	43
17. Endomorphisms, automorphisms, center of a group	45
18. Conjugate classes	47

CHAPTER II: RINGS, INTEGRAL DOMAINS AND FIELDS	
SECTION	PAGE
1. Definition and examples	49
2. Types of rings	53
3. Quasi-regularity. The circle composition	55
4. Matrix rings	56
5. Quaternions	60
6. Subrings generated by a set of elements. Center	63
7. Ideals, difference rings	64
8. Ideals and difference rings for the ring of integers	66
9. Homomorphism of rings	68
10. Anti-isomorphism	71
11. Structure of the additive group of a ring. The characteristic of a ring	74
12. Algebra of subgroups of the additive group of a ring. One- sided ideals	75
13. The ring of endomorphisms of a commutative group	78
14. The multiplications of a ring	82
CHAPTER III: EXTENSIONS OF RINGS AND FIELDS	
1. Imbedding of a ring in a ring with an identity	84
2. Field of fractions of a commutative integral domain	87
3. Uniqueness of the field of fractions	91
4. Polynomial rings	92
5. Structure of polynomial rings	96
6. Properties of the ring $\mathfrak{A}[x]$	97
7. Simple extensions of a field	100
8. Structure of any field	103
9. The number of roots of a polynomial in a field	104
10. Polynomials in several elements	105
11. Symmetric polynomials	107
12. Rings of functions	110
CHAPTER IV: ELEMENTARY FACTORIZATION THEORY	
1. Factors, associates, irreducible elements	114
2. Gaussian semi-groups	115
3. Greatest common divisors	118
4. Principal ideal domains	121

SECTION	PAGE
5. Euclidean domains	122
6. Polynomial extensions of Gaussian domains	124

CHAPTER V: GROUPS WITH OPERATORS

1. Definition and examples of groups with operators	128
2. M-subgroups, M-factor groups and M-homomorphisms	130
3. The fundamental theorem of homomorphism for M-groups	132
4. The correspondence between M-subgroups determined by a homomorphism	133
5. The isomorphism theorems for M-groups	135
6. Schreier's theorem	137
7. Simple groups and the Jordan-Hölder theorem	139
8. The chain conditions	142
9. Direct products	144
10. Direct products of subgroups	145
11. Projections	149
12. Decomposition into indecomposable groups	152
13. The Krull-Schmidt theorem	154
14. Infinite direct products	159

CHAPTER VI: MODULES AND IDEALS

1. Definitions	162
2. Fundamental concepts	164
3. Generators. Unitary modules	166
4. The chain conditions	168
5. The Hilbert basis theorem	170
6. Noetherian rings. Prime and primary ideals	172
7. Representation of an ideal as intersection of primary ideals	175
8. Uniqueness theorems	177
9. Integral dependence	181
10. Integers of quadratic fields	184

CHAPTER VII: LATTICES

1. Partially ordered sets	187
2. Lattices	189
3. Modular lattices	193
4. Schreier's theorem. The chain conditions	197

SECTION	PAGE
5. Decomposition theory for lattices with ascending chain condition	201
6. Independence	202
7. Complemented modular lattices	205
8. Boolean algebras	207
Index	213