

Finite-Dimensional Division Algebras over Fields

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PREFACE

These algebras determine, by the Wedderburn Theorem, the semi-simple finite dimensional algebras over a field. They lead to the definition of the Brauer group and to certain geometric objects, the Brauer-Severi varieties.

We shall be interested in these algebras which have an involution. Algebras with involution arose first in the study of the so-called "multiplication algebras of Riemann matrices". Albert undertook their study at the behest of Lefschetz. He solved the problem of determining these algebras. The problem has an algebraic part and an arithmetic part which can be solved only by determining the finite dimensional simple algebras over an algebraic number field. We are not going to consider the arithmetic part but will be interested only in the algebraic part. In Albert's classical book (1939), both parts are treated. A quick survey of our Table of Contents will indicate the scope of the present volume.

The largest part of our book is the fifth chapter which deals with involutorial simple algebras of finite dimension over a field.

Of particular interest are the Jordan algebras determined by these algebras with involution. Their structure is determined and two important concepts of these algebras with involution are the universal enveloping algebras and the reduced norm. Of great importance is the concept of isotopy. There are numerous applications of these concepts, some of which are quite old.

In preparing this volume we have been assisted by our friends, notably Jean-Pierre Tignol and John Faulkner. Also, I am greatly indebted to my secretary, Donna Belli, and to my wife, Florie. I wish to thank all of them for their help.

Table of Contents

I. Skew Polynomials and Division Algebras.....	1
1.1. Skew-polynomial Rings.....	1
1.2. Arithmetic in a PID.....	10
1.3. Applications to Skew-polynomial Rings.....	14
1.4. Cyclic and Generalized Cyclic Algebras.....	19
1.5. Generalized Differential Extensions.....	21
1.6. Reduced Characteristic Polynomial, Trace and Norm.....	24
1.7. Norm Conditions.....	29
1.8. Derivations of Purely Inseparable Extensions of Exponent One....	31
1.9. Some Tensor Product Constructions.....	33
1.10. Twisted Laurent Series.....	37
1.11. Differential Laurent Series.....	38
II. Brauer Factor Sets and Noether Factor Sets.....	41
2.1. Frobenius Algebras.....	42
2.2. Commutative Frobenius Subalgebras.....	43
2.3. Brauer Factor Sets.....	45
2.4. Condition for Split Algebra. The Tensor Product Theorem.....	51
2.5. The Brauer Group $\text{Br}(K/F)$	53
2.6. Crossed Products.....	56
2.7. The Exponent of a Central Simple Algebra.....	60
2.8. Central Division Algebras of Prescribed Exponent and Degree....	62
2.9. Central Division Algebras of Degree ≤ 4	66
2.10. Non-cyclic Division Algebras of Degree Four.....	76
2.11. A Criterion for Cyclicity of a Division Algebra of Prime Degree...	80
2.12. Central Division Algebras of Degree Five.....	84
2.13. Inflation and Restriction for Crossed Products.....	86
2.14. Isomorphism of $\text{Br}(F)$ and $H^2(F)$	91
III. Galois Descent and Generic Splitting Fields.....	95
3.1. Galois Descent for Vector Spaces.....	96
3.2. Forms of Fields.....	99
3.3. Forms and Non-commutative Cohomology.....	102
3.4. Grassmannians.....	107
3.5. Brauer-Severi Varieties.....	111

3.6. Properties of Brauer-Severi Varieties	114
3.7. Brauer-Severi Varieties and Brauer Fields	118
3.8. Generic Splitting Fields	123
3.9. Properties of Brauer Fields	125
3.10. Central Simple Algebras Split by a Brauer Field	130
3.11. Norm Hypersurface of a Central Simple Algebra	138
3.12. Variety of Rank One Elements	140
3.13. The Brauer Functor. Corestriction of Algebras	149
IV. p-Algebras	154
4.1. The Frobenius Map and Purely Inseparable Splitting Fields	155
4.2. Similarity to Tensor Products of Cyclic Algebras	158
4.3. Galois Extensions of Prime Power Degree	162
4.4. Conditions for Cyclicity	166
4.5. Similarity to Cyclic Algebras	171
4.6. Generic Abelian Crossed Products	174
4.7. Non-cyclic p-Algebras	182
V. Simple Algebras with Involution	185
5.1. Generalities. Simple Algebras with Involution	186
5.2. Existence of Involutions in Simple Algebras	193
5.3. Reduced Norms of Special Jordan Algebras	197
5.4. Differential Calculus of Rational Maps	204
5.5. Basic Properties of Reduced Norms	205
5.6. Low Dimensional Involutorial Division Algebras. Positive Results	209
5.7. Some Counterexamples	221
5.8. Decomposition of Simple Algebras with Involution of Degree 4 ...	232
5.9. Multiplicative Properties of Reduced Norms	235
5.10. Isotopy and Norm Similarity	240
5.11. Special Universal Envelopes	247
5.12. Applications to Norm Similarities	256
5.13. The Jordan Algebra $H(A, J)$	262
References	275