

Kiyosi Itô

# Stochastic Processes

Lectures given at Aarhus University

Edited by  
Ole E. Barndorff-Nielsen  
Ken-iti Sato



Springer

*Kiyosi Itô*  
Kyoto, Japan

*Ole E. Barndorff-Nielsen*  
Department of Mathematics  
University of Aarhus  
Ny Munkegade  
8000 Aarhus, Denmark  
e-mail: oebn@imf.au.dk

*Ken-iti Sato*  
Hachiman-yama 1101-5-103  
Tenpaku-ku, Japan  
e-mail: ken-iti.sato@nifty.ne.jp

Cataloging-in-Publication Data applied for

A catalog record for this book is available from the Library of Congress.

Bibliographic information published by Die Deutsche Bibliothek

Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie;  
detailed bibliographic data is available in the Internet at <<http://dnb.ddb.de>>.

Mathematics Subject Classification (2000): 60-02, 60E07, 60G51, 60J25

ISBN 978-3-642-05805-9 ISBN 978-3-662-10065-3 (eBook)

DOI 10.1007/978-3-662-10065-3

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 2004

Originally published by Springer-Verlag Berlin Heidelberg New York in 2004.

Softcover reprint of the hardcover 1st edition 2004

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Typeset by the editors.

Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper 41/3142LK - 5 4 3 2 1 0

# Table of Contents

<b>0</b>	<b>Preliminaries</b> .....	1
0.1	Independence .....	1
0.2	Central Values and Dispersions .....	5
0.3	Centralized Sum of Independent Random Variables .....	12
0.4	Infinitely Divisible Distributions .....	18
0.5	Continuity and Discontinuity of Infinitely Divisible Distributions .....	25
0.6	Conditional Probability and Expectation .....	27
0.7	Martingales .....	32
<b>1</b>	<b>Additive Processes</b>	
	<b>(Processes with Independent Increments)</b> .....	39
1.1	Definitions .....	39
1.2	Decomposition of Additive Processes .....	41
1.3	The Lévy Modification of Additive Processes Continuous in Probability .....	45
1.4	Elementary Lévy Processes .....	50
1.5	Fundamental Lemma .....	58
1.6	Structure of Sample Functions of Lévy Processes (a) .....	61
1.7	Structure of Sample Functions of Lévy Processes (b) .....	68
1.8	Three Components of Lévy Processes .....	74
1.9	Random Point Measures .....	77
1.10	Homogeneous Additive Processes and Homogeneous Lévy Processes .....	83
1.11	Lévy Processes with Increasing Paths .....	85
1.12	Stable Processes .....	88
<b>2</b>	<b>Markov Processes</b> .....	93
2.1	Transition Probabilities and Transition Operators on Compact Metrizable Spaces .....	93
2.2	Summary of the Hille–Yosida Theory of Semi-Groups .....	95
2.3	Transition Semi-Group .....	101
2.4	Probability Law of the Path .....	103

2.5	Markov Property .....	110
2.6	The $\sigma$ -Algebras $\overline{\mathcal{B}}$ , $\overline{\mathcal{B}}_t$ , and $\overline{\mathcal{B}}(S)$ .....	115
2.7	Strong Markov Property .....	118
2.8	Superposition of Stopping Times .....	123
2.9	An Inequality of Kolmogorov Type and its Application .....	124
2.10	Hitting Times of Closed Sets .....	129
2.11	Dynkin's Formula .....	130
2.12	Markov Processes in Generalized Sense .....	135
2.13	Examples .....	143
2.14	Markov Processes with a Countable State Space .....	148
2.15	Fine Topology .....	154
2.16	Generator in Generalized Sense .....	159
2.17	The Kac Semi-Group and its Application to the Arcsine Law .....	164
2.18	Markov Processes and Potential Theory .....	170
2.19	Brownian Motion and the Dirichlet Problem .....	172
<b>Exercises</b> .....		179
E.0	Chapter 0 .....	179
E.1	Chapter 1 .....	188
E.2	Chapter 2 .....	190
<b>Appendix: Solutions of Exercises</b> .....		193
A.0	Chapter 0 .....	193
A.1	Chapter 1 .....	215
A.2	Chapter 2 .....	221
<b>Index</b> .....		231