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Saint-Venant's Problem

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INTRODUCTION

A major concern throughout the history of elasticity has been with problems dictated by the demands of engineering. Interest in the construction of a theory for the deformation of elastic cylinders dates back to Coulomb, Navier and Cauchy. However, only Saint-Venant has been able to give a solution of the problem.

The importance of Saint-Venant's celebrated memoirs [132,133], on what has long since become known as Saint-Venant's problem requires no emphasis. To review the vast literature to which the work contained in [132,133] has given impetus is not our intention. An account of the historical developments as well as references to various contributions, may be found in the books and in some of the papers cited.

We recall that Saint-Venant's problem consists in determining the equilibrium of a homogeneous and isotropic linearly elastic cylinder, loaded by surface forces distributed over its plane ends. Saint-Venant proposed an approximation to the solution of the three-dimensional problem, which only requires the solution of two-dimensional problems in the cross section of the cylinder. Saint-Venant's formulation leads to the four basic problems of extension, bending, torsion and flexure. His analysis is founded on physical intuition and elementary beam theory. Saint-Venant's extension, bending, torsion, and flexure solutions are well-known (see, for example, Love [98], Chapters 14,15 and Sokolnikoff [139], Chapter 4).

Saint-Venant's approach of the problem is based on a relaxed statement in which the pointwise assignment of the terminal tractions is replaced by prescribing the corresponding resultant force and resultant moment. Justification of the procedure is twofold. First, it is difficult in practice to determine the actual distribution of applied stresses on the ends, although the resultant force and moment can be measured accurately. Second, one invokes Saint-Venant's principle. This principle states, roughly speaking, that if two sets of loadings are statically equivalent at each end, then the difference in stress fields and strain fields are negligible, except possibly near the ends. The precise meaning of Saint-Venant's hypothesis and its justification have been the subject of many studies, almost from the time of the original Saint-Venant's papers. Reference to some of the early investigations of the question will be found in [98],[139],[140]. In recent years important steps toward clarifying Saint-Venant's principle have been made. The classic paper in linear elasticity is by Toupin [146] (see also, e.g. Roseman [128], Knowles [84] and Fichera [39,40] for further important developments). For the history of the problem and the detailed analysis of various results on Saint-Venant's principle we refer to the works of M.E.Gurtin [47], G.Fichera [38], C.O.Horgan and J.K.Knowles [53].

The relaxed Saint-Venant's problem continues to attract attention both from the mathematical and the technical point of view.

It is obvious that the relaxed statement of the problem fails to characterize the solution uniquely. This fact led various authors to establish characterizations of Saint-Venant's solution. Thus, Clebsch [24] proved that Saint-Venant's solution can be derived from the assumption that the stress vector on any plane normal to the cross-sections of the cylinder is parallel to its generators. In [155], Voigt rediscovered Saint-Venant's solution by using another assumption regarding the structure of the stress field. Thus, Saint-Venant's extension, bending and torsion solutions are derived from the hypothesis that the stress field is independent of the axial coordinate, and Saint-Venant's flexure solution is obtained if the stress field depends on the axial coordinate at most linearly.

E.Sternberg and J.K.Knowles [143] characterized Saint-Venant's solutions in terms of certain associated minimum strain-energy properties. Other intrinsic criteria that distinguish Saint-Venant's solutions among all the solutions of the relaxed problem were established in [79]. In [79], a rational scheme of deriving Saint-Venant's solutions is presented. The advantage of this method is that it does not involve artificial a priori assumptions. The method permits to construct a solution of the relaxed Saint-Venant's problem for other kinds of constitutive equations (anisotropic media, Cosserat continua, etc.) where the physical intuition or semi-inverse method cannot be used.

In [148]-[150], C.Truesdell proposed a problem which, roughly speaking, consists in the generalization of Saint-Venant's notion of twist so as to apply to any solution of the torsion problem. Recently an elegant solution of Truesdell's problem has been established by W.A.Day [25]. In [123], P.Podio-Guidugli studied Truesdell's problem rephrased for extension and pure bending. The case of flexure was considered in [79]. The results of [25,123] are related to the results of Sternberg and Knowles [143] concerning the minimum energy characterizations of corresponding Saint-Venant's solutions.

A generalization of the relaxed Saint-Venant's problem consists in determining the equilibrium of an elastic cylinder which - in the presence of body forces - is subjected to surface tractions arbitrarily prescribed over the lateral boundary and to appropriate stress resultants over its ends. The study of this problem was initiated by Almansi [1] and Michell [102] and was developed in various later papers (see,for example, Sokolnikoff [139], Djanelidze [29] and Hatiashvili [49]).

As pointed out before, Saint-Venant's results were established within the equilibrium theory of homogeneous and isotropic elastic bodies. A large number of papers are concerned with the relaxed Saint-Venant's problem for other kinds of elastic materials (see, for example, Lekhnitskii [96], Lomakin [97], Brulin and R.K.T.Hsieh [15] and Reddy and Venkatasubramanian [90]). References to recent results are cited in the text. No attempt is made to provide a complete list of works on Saint-Venant's problem. Neither the contents, nor the list of works cited are exhaustive. Nevertheless, it is hoped that the developments presented reflect the state of knowledge in the study of the problem.

The purpose of this work is to present some of the recent researches on Saint-Venant's problem. An effort is made to provide a systematic treatment of the subject.

Chapter 1 is concerned mainly with results where Saint-Venant's solutions are involved. We give a rational method of construction of these solutions and then we characterize them in terms of certain associated minimum strain-energy properties. A study of Truesdell's problem is presented. This chapter also includes a proof of Saint-Venant's principle.

In Chapter 2, an interesting scheme of deriving a solution of

Almansi-Michell problem is presented. Almansi's problem, where the body forces and the surface tractions on the lateral boundary are polynomials in the axial coordinate, is also studied. The results are used to study a statical problem in the linear thermoelasticity.

Chapter 3 is concerned with the relaxed Saint-Venant's problem for anisotropic elastic bodies. We first establish a solution of the foregoing problem. The method does not involve artificial a priori assumptions and permits a treatment of the problem even for nonhomogeneous bodies, where the elastic coefficients are independent of the axial coordinate. It is shown that the well-known boundary-value problem for the torsion function derives from a special problem of generalized plane strain. Then, minimum energy characterizations of the solutions are presented. Also included in this chapter is a study of Truesdell's problem.

Chapter 4 deals with the relaxed Saint-Venant's problem for heterogeneous elastic cylinders. We consider the case of a composed cylinder when the generic cross-section is occupied by different anisotropic solids. The problems of Almansi and Michell are also studied. Applications to the linear thermoelastostatics are given.

In Chapter 5 we study Saint-Venant's problem within the linearized theory of Cosserat elastic bodies. We first present a proof of Saint-Venant's principle in the theory of Cosserat elasticity. Then, a solution of the relaxed Saint-Venant's problem is derived. Truesdell's problem and a theory of loaded cylinders are also studied. Illustrative applications are presented.

A number of results included in this work have not appeared or been discussed previously in literature.

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