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Polyharmonic Boundary Value Problems

Positivity Preserving and Nonlinear Higher
Order Elliptic Equations in Bounded Domains

Contents

Preface.....	vii
Acknowledgements.....	xiii
1 Models of Higher Order.....	1
1.1 Classical Problems from Elasticity	1
1.1.1 The Static Loading of a Slender Beam.....	2
1.1.2 The Kirchhoff-Love Model for a Thin Plate.....	5
1.1.3 Decomposition into Second Order Systems.....	8
1.2 The Boggio-Hadamard Conjecture for a Clamped Plate.....	9
1.3 The First Eigenvalue.....	12
1.3.1 The Dirichlet Eigenvalue Problem	13
1.3.2 An Eigenvalue Problem for a Buckled Plate.....	14
1.3.3 A Steklov Eigenvalue Problem.....	15
1.4 Paradoxes for the Hinged Plate	16
1.4.1 Sapondžyan's Paradox by Concave Corners.....	17
1.4.2 The Babuška Paradox	17
1.5 Paneitz-Branson Type Equations.....	18
1.6 Critical Growth Polyharmonic Model Problems	21
1.7 Qualitative Properties of Solutions to Semilinear Problems.....	22
1.8 Willmore Surfaces	23
2 Linear Problems	27
2.1 Polyharmonic Operators.....	27
2.2 Higher Order Sobolev Spaces.....	29
2.2.1 Definitions and Basic Properties	29
2.2.2 Embedding Theorems.....	32
2.3 Boundary Conditions	33
2.4 Hilbert Space Theory	36
2.4.1 Normal Boundary Conditions and Green's Formula	36
2.4.2 Homogeneous Boundary Value Problems	38
2.4.3 Inhomogeneous Boundary Value Problems.....	42

2.5	Regularity Results and A Priori Estimates	44
2.5.1	Schauder Theory	44
2.5.2	L^p -Theory	46
2.5.3	The Miranda-Agmon Maximum Modulus Estimates	48
2.6	Green's Function and Boggio's Formula	50
2.7	The Space $H^2 \cap H_0^1$ and the Sazonov-Babuška Paradoxes	52
2.8	Bibliographical Notes	59
3	Eigenvalue Problems	61
3.1	Dirichlet Eigenvalues	62
3.1.1	A Generalised Krein-Rutman Result	62
3.1.2	Decomposition with Respect to Dual Cones	63
3.1.3	Positivity of the First Eigenfunction	68
3.1.4	Symmetrisation and Talenti's Comparison Principle	71
3.1.5	The Rayleigh Conjecture for the Clamped Plate	73
3.2	Buckling Load of a Clamped Plate	77
3.3	Steklov Eigenvalues	82
3.3.1	The Steklov Spectrum	83
3.3.2	Minimisation of the First Eigenvalue	90
3.4	Bibliographical Notes	96
4	Kernel Estimates	99
4.1	Consequences of Boggio's Formula	99
4.2	Kernel Estimates in the Ball	101
4.2.1	Direct Green Function Estimates	101
4.2.2	A 3-G-Type Theorem	110
4.3	Estimates for the Steklov Problem	116
4.4	General Properties of the Green Functions	122
4.4.1	Regularity of the Biharmonic Green Function	122
4.4.2	Preliminary Estimates for the Green Function	123
4.5	Uniform Green Functions Estimates in $C^{k,\gamma}$ -Families of Domains	125
4.5.1	Uniform Global Estimates Without Boundary Terms	126
4.5.2	Uniform Global Estimates Including Boundary Terms	135
4.5.3	Convergence of the Green Function in Domain Approximations	141
4.6	Weighted Estimates for the Dirichlet Problem	142
4.7	Bibliographical Notes	146
5	Positivity and Lower Order Perturbations	147
5.1	A Positivity Result for Dirichlet Problems in the Ball	148
5.2	The Role of Positive Boundary Data	153
5.2.1	The Highest Order Dirichlet Datum	153
5.2.2	Also Nonzero Lower Order Boundary Terms	157
5.3	Local Maximum Principles for Higher Order Differential Inequalities	165

5.4	Steklov Boundary Conditions	167
5.4.1	Positivity Preserving	168
5.4.2	Positivity of the Operators Involved in the Steklov Problem	175
5.4.3	Relation Between Hilbert and Schauder Setting	178
5.5	Bibliographical Notes	185
6	Dominance of Positivity in Linear Equations	187
6.1	Highest Order Perturbations in Two Dimensions	188
6.1.1	Domain Perturbations	190
6.1.2	Perturbations of the Principal Part	193
6.2	Small Negative Part of Biharmonic Green's Functions in Two Dimensions	197
6.2.1	The Biharmonic Green Function on the Limaçons de Pascal	197
6.2.2	Filling Smooth Domains with Perturbed Limaçons	201
6.3	Regions of Positivity in Arbitrary Domains in Higher Dimensions	208
6.3.1	The Biharmonic Operator	210
6.3.2	Extensions to Polyharmonic Operators	215
6.4	Small Negative Part of Biharmonic Green's Functions in Higher Dimensions	216
6.4.1	Bounds for the Negative Part	217
6.4.2	A Blow-Up Procedure	218
6.5	Domain Perturbations in Higher Dimensions	224
6.6	Bibliographical Notes	226
7	Semilinear Problems	227
7.1	A Gidas-Ni-Nirenberg Type Symmetry Result	229
7.1.1	Green Function Inequalities	231
7.1.2	The Moving Plane Argument	234
7.2	A Brief Overview of Subcritical Problems	238
7.2.1	Regularity for At Most Critical Growth Problems	238
7.2.2	Existence	241
7.2.3	Positivity and Symmetry	243
7.3	The Hilbertian Critical Embedding	244
7.4	The Pohožaev Identity for Critical Growth Problems	254
7.5	Critical Growth Dirichlet Problems	260
7.5.1	Nonexistence Results	260
7.5.2	Existence Results for Linearly Perturbed Equations	263
7.5.3	Nontrivial Solutions Beyond the First Eigenvalue	271
7.6	Critical Growth Navier Problems	281
7.7	Critical Growth Steklov Problems	285
7.8	Optimal Sobolev Inequalities with Remainder Terms	295

7.9	Critical Growth Problems in Geometrically Complicated Domains	300
7.9.1	Existence Results in Domains with Nontrivial Topology	301
7.9.2	Existence Results in Contractible Domains	302
7.9.3	Energy of Nodal Solutions	304
7.9.4	The Deformation Argument	307
7.9.5	A Struwe-Type Compactness Result	311
7.10	The Conformally Covariant Paneitz Equation in Hyperbolic Space	319
7.10.1	Infinitely Many Complete Radial Conformal Metrics with the Same Q-Curvature	319
7.10.2	Existence and Negative Scalar Curvature	320
7.10.3	Completeness of the Conformal Metric	326
7.11	Fourth Order Equations with Supercritical Terms	335
7.11.1	An Autonomous System	339
7.11.2	Regular Minimal Solutions	347
7.11.3	Characterisation of Singular Solutions	352
7.11.4	Stability of the Minimal Regular Solution	357
7.11.5	Existence and Uniqueness of a Singular Solution	360
7.12	Bibliographical Notes	366
8	Willmore Surfaces of Revolution	371
8.1	An Existence Result	371
8.2	Geometric Background	372
8.2.1	Geometric Quantities for Surfaces of Revolution	372
8.2.2	Surfaces of Revolution as Elastic Curves in the Hyperbolic Half Plane	376
8.3	Minimisation of the Willmore Functional	381
8.3.1	An Upper Bound for the Optimal Energy	382
8.3.2	Monotonicity of the Optimal Energy	383
8.3.3	Properties of Minimising Sequences	387
8.3.4	Attainment of the Minimal Energy	389
8.4	Bibliographical Notes	392
	Notations	393
	References	397
	Author-Index	415
	Subject-Index	419