

Michel Broué

Introduction to Complex Reflection Groups and Their Braid Groups

Contents

| | | |
|----------|--|-----------|
| 1 | Preliminaries | 1 |
| 1.1 | Reflections and Roots | 1 |
| 1.1.1 | Rank One Endomorphisms | 1 |
| 1.1.2 | Projections, Transvections, Reflections | 2 |
| 1.1.3 | Reflections | 2 |
| 1.1.4 | Commuting Reflections | 3 |
| 1.2 | Reflection Groups | 4 |
| 1.2.1 | Orthogonal Decomposition | 5 |
| 1.2.2 | The Shephard–Todd Classification | 6 |
| 1.2.3 | Reflecting Pairs | 8 |
| 2 | Prerequisites and Complements in Commutative Algebra | 11 |
| 2.1 | Finite Ring Extensions | 11 |
| 2.1.1 | Properties and Definitions | 11 |
| 2.1.2 | Spectra and Finite Extensions | 12 |
| 2.1.3 | Case of Integrally Closed Rings | 13 |
| 2.1.4 | Krull Dimension: First Definitions | 13 |
| 2.2 | Jacobson Rings and Hilbert's Nullstellensatz | 15 |
| 2.2.1 | On Maximal Ideal of Polynomial Algebras | 15 |
| 2.2.2 | Radicals and Jacobson Rings, Application to Algebraic Varieties | 19 |
| 2.3 | Graded Algebras and Modules | 20 |
| 2.3.1 | Graded Modules | 20 |
| 2.3.2 | Elementary Constructions | 21 |
| 2.3.3 | Kozenil Complex | 22 |
| 2.3.4 | Graded Algebras and Modules | 23 |
| 2.3.5 | The Hilbert–Serre Theorem | 29 |
| 2.3.6 | Nakayama's Lemma | 24 |
| 2.4 | Polynomial Algebras and Parameters Subalgebras | 26 |
| 2.4.1 | Degrees and Jacobian | 26 |
| 2.4.2 | Systems of Parameters | 28 |
| 2.4.3 | The Chevalley Theorem | 31 |

| | |
|---|----|
| 3 Polynomial Invariants of Finite Linear Groups | 35 |
| 3.1 Finite Group Invariants | 35 |
| 3.1.1 Generalities | 35 |
| 3.1.2 Case of Height One Primes | 37 |
| 3.2 Finite Linear Groups on Symmetric Algebras | 38 |
| 3.2.1 Ramification and Reflecting Pairs | 39 |
| 3.2.2 Linear Characters Associated with Reflecting Hyperplanes | 40 |
| 3.3 Coinvariant Algebra and Harmonic Polynomials | 44 |
| 3.3.1 The Coinvariant Algebra | 44 |
| 3.3.2 Galois Twisting of a Representation | 45 |
| 3.3.3 Differential Operators, Harmonic Polynomials | 46 |
| 3.4 Graded Characters and Applications | 48 |
| 3.4.1 Graded Characters of Graded kG -Modules | 49 |
| 3.4.2 Isotypic Components of the Symmetric Algebra | 50 |
| 3.4.3 Some Numerical Identities | 50 |
| 3.4.4 Isotypic Components Are Cohen–Macaulay | 52 |
| 3.4.5 Computations with Power Series | 52 |
| 3.4.6 A Simple Example | 55 |
| 4 Finite Reflection Groups in Characteristic Zero | 57 |
| 4.1 The Shephard–Todd/Chevalley–Serre Theorem | 57 |
| 4.2 Steinberg Theorem and First Applications | 60 |
| 4.2.1 The Jacobian as a Monomial | 60 |
| 4.2.2 Action of the Normalizer and Generalized Degrees | 60 |
| 4.2.3 Steinberg Theorem | 62 |
| 4.2.4 Fixed Points of Elements of G | 64 |
| 4.2.5 Braid Groups | 65 |
| 4.3 Coinvariant Algebras and Harmonic Polynomials | 67 |
| 4.3.1 On the Coinvariant Algebra | 67 |
| 4.3.2 Linear Characters and Their Associated Polynomials | 69 |
| 4.3.3 The Harmonic Elements of a Reflection Group and the Poincaré Duality | 72 |
| 4.4 Application to Braid Groups | 75 |
| 4.4.1 Discriminants and Length | 75 |
| 4.4.2 Complement: Artin-Like Presentations of the Braid Diagrams | 77 |
| 4.5 Graded Multiplicities and Solomon’s Theorem | 78 |
| 4.5.1 Preliminary: Graded Dimension of $(S \otimes V)^G$ | 78 |
| 4.5.2 Exponents and Gutkin–Opdam Matrices | 80 |
| 4.5.3 Solomon Theorem | 89 |
| 4.5.4 Derivations and Differential Forms on V | 91 |
| 4.5.5 First Applications of Solomon’s Theorem | 94 |

| | |
|---|-----|
| Contents | x1 |
| 5 Eigenspaces and Regular Elements | 97 |
| 5.1 Eigenspaces | 97 |
| 5.1.1 Pianzola–Weis Formula | 97 |
| 5.1.2 Maximal Eigenspaces: Lehrer–Springer Theory | 99 |
| 5.2 Regular Elements | 104 |
| 5.2.1 First Properties | 104 |
| 5.2.2 Exponents and Eigenvalues of Regular Elements | 106 |
| 5.3 Regular Braids Automorphisms | 112 |
| 5.3.1 Lifting Regular Automorphisms: Case When the Base Point Is an Eigenvector | 112 |
| 5.3.2 Lifting Regular Automorphisms: General Case | 115 |
| 5.3.3 Lifting Springer Theory | 117 |
| Coxeter and Artin–Like Presentations | 119 |
| A Coxeter and Artin–Like Presentations | 119 |
| A.1 Meaning of the Diagrams | 119 |
| A.1.1 Diagrams for the Reflection Groups | 119 |
| A.1.2 Braid Diagrams | 121 |
| A.2 Tables | 124 |
| References | 133 |
| Index | 137 |