## Anthony W. Knapp

# Advanced Real Analysis

Along with a companion volume *Basic Real Analysis* 

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