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Calculus of Variations I

With 73 Figures



Springer

Contents of Calculus of Variations I

The Lagrangian Formalism

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