

Marc Bernot · Vicent Caselles
Jean-Michel Morel

Optimal Transportation Networks

Models and Theory

 Springer

Table of Contents

1	Introduction: The Models	1
2	The Mathematical Models	11
2.1	The Monge-Kantorovich Problem	11
2.2	The Gilbert-Steiner Problem	12
2.3	Three Continuous Extensions of the Gilbert-Steiner Problem	13
2.3.1	Xia's Transport Paths	13
2.3.2	Maddalena-Solimini's Patterns	14
2.3.3	Traffic Plans	14
2.4	Questions and Answers	16
2.4.1	Plan	17
2.5	Related Problems and Models	19
2.5.1	Measures on Sets of Paths	19
2.5.2	Urban Transportation Models with more than One Transportation Means	20
3	Traffic Plans	25
3.1	Parameterized Traffic Plans	27
3.2	Stability Properties of Traffic Plans	29
3.2.1	Lower Semicontinuity of Length, Stopping Time, Averaged Length and Averaged Stopping Time	30
3.2.2	Multiplicity of a Traffic Plan and its Upper Semicontinuity	31
3.2.3	Sequential Compactness of Traffic Plans	33
3.3	Application to the Monge-Kantorovich Problem	34
3.4	Energy of a Traffic Plan and Existence of a Minimizer	35
4	The Structure of Optimal Traffic Plans	39
4.1	Speed Normalization	39
4.2	Loop-Free Traffic Plans	41
4.3	The Generalized Gilbert Energy	42
4.3.1	Rectifiability of Traffic Plans with Finite Energy	44
4.4	Appendix: Measurability Lemmas	44

5	Operations on Traffic Plans	47
5.1	Elementary Operations	47
5.1.1	Restriction, Domain of a Traffic Plan	47
5.1.2	Sum of Traffic Plans (or Union of their Parameterizations)	48
5.1.3	Mass Normalization	48
5.2	Concatenation	48
5.2.1	Concatenation of Two Traffic Plans	48
5.2.2	Hierarchical Concatenation (Construction of Infinite Irrigating Trees or <i>Patterns</i>)	49
5.3	A Priori Properties on Minimizers	51
5.3.1	An Assumption on μ^+ , μ^- and π Avoiding Fibers with Zero Length	51
5.3.2	A Convex Hull Property	53
6	Traffic Plans and Distances between Measures	55
6.1	All Measures can be Irrigated for $\alpha > 1 - \frac{1}{N}$	56
6.2	Stability with Respect to μ^+ and μ^-	58
6.3	Comparison of Distances between Measures	59
7	The Tree Structure of Optimal Traffic Plans and their Approximation	65
7.1	The Single Path Property	65
7.2	The Tree Property	70
7.3	Decomposition into Trees and Finite Graphs Approximation	71
7.4	Bi-Lipschitz Regularity	77
8	Interior and Boundary Regularity	79
8.1	Connected Components of a Traffic Plan	79
8.2	Cuts and Branching Points of a Traffic Plan	81
8.3	Interior Regularity	82
8.3.1	The Main Lemma	82
8.3.2	Interior Regularity when $\overline{\text{supp}(\mu^+)} \cap \overline{\text{supp}(\mu^-)} = \emptyset$...	85
8.3.3	Interior Regularity when μ^+ is a Finite Atomic Measure	89
8.4	Boundary Regularity	91
8.4.1	Further Regularity Properties	93
9	The Equivalence of Various Models	95
9.1	Irrigating Finite Atomic Measures (Gilbert-Steiner) and Traffic Plans	95
9.2	Patterns (Maddalena et al.) and Traffic Plans	96
9.3	Transport Paths (Qinglan Xia) and Traffic Plans	97
9.4	Optimal Transportation Networks as Flat Chains	100

10 Irrigability and Dimension	105
10.1 Several Concepts of Dimension of a Measure and Irrigability Results	105
10.2 Lower Bound on $d(\mu)$	111
10.3 Upper Bound on $d(\mu)$	112
10.4 Remarks and Examples	114
11 The Landscape of an Optimal Pattern	119
11.1 Introduction	119
11.1.1 Landscape Equilibrium and OCNs in Geophysics	119
11.2 A General Development Formula	122
11.3 Existence of the Landscape Function and Applications	124
11.3.1 Well-Definedness of the Landscape Function	124
11.3.2 Variational Applications	127
11.4 Properties of the Landscape Function	128
11.4.1 Semicontinuity	128
11.4.2 Maximal Slope in the Network Direction	129
11.5 Hölder Continuity under Extra Assumptions	131
11.5.1 Campanato Spaces by Medians	131
11.5.2 Hölder Continuity of the Landscape Function	132
12 The Gilbert-Steiner Problem	135
12.1 Optimum Irrigation from One Source to Two Sinks	135
12.2 Optimal Shape of a Traffic Plan with given Dyadic Topology	143
12.2.1 Topology of a Graph	143
12.2.2 A Recursive Construction of an Optimum with Full Steiner Topology	144
12.3 Number of Branches at a Bifurcation	145
13 Dirac to Lebesgue Segment: A Case Study	151
13.1 Analytical Results	152
13.1.1 The Case of a Source Aligned with the Segment	152
13.2 A "T Structure" is not Optimal	153
13.3 The Boundary Behavior of an Optimal Solution	155
13.4 Can Fibers Move along the Segment in the Optimal Structure?	159
13.5 Numerical Results	159
13.5.1 Coding of the Topology	159
13.5.2 Exhaustive Search	160
13.6 Heuristics for Topology Optimization	160
13.6.1 Multiscale Method	161
13.6.2 Optimality of Subtrees	164
13.6.3 Perturbation of the Topology	165

14 Application: Embedded Irrigation Networks	169
14.1 Irrigation Networks made of Tubes	169
14.1.1 Anticipating some Conclusions	171
14.2 Getting Back to the Gilbert Functional	172
14.3 A Consequence of the Space-filling Condition	175
14.4 Source to Volume Transfer Energy	176
14.5 Final Remarks	177
15 Open Problems	179
15.1 Stability	179
15.2 Regularity	179
15.3 The who goes where Problem	180
15.4 Dirac to Lebesgue Segment	180
15.5 Algorithm or Construction of Local Optima	181
15.6 Structure	182
15.7 Scaling Laws	183
15.8 Local Optimality in the Case of Non Irrigability	183
A Skorokhod Theorem	185
B Flows in Tubes	189
B.1 Poiseuille's Law	189
B.2 Optimality of the Circular Section	190
C Notations	191
References	193
Index	199