
MATRIX-BASED MULTIGRID

Theory and Applications

Second Edition

YAIR SHAPIRA

Technion-Israel Institute of Technology



Springer

Yair Shapira
Technion – Israel Institute of Technology
Department of Computer Science
32000 Haifa, Israel
yairs@cs.technion.ac.il

ISBN: 978-0-387-49764-8 e-ISBN: 978-0-387-49765-5
DOI: 10.1007/978-0-387-49765-5

ISSN: 1571-5698

Library of Congress Control Number: 2007934905

AMS Subject Classifications: 65N55, 65F10

© 2008 Springer Science+Business Media, LLC

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer Science+Business Media, LLC, 233 Spring Street, New York, NY 10013, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use in this publication of trade names, trademarks, service marks, and similar terms, even if they are not identified as such, is not to be taken as an expression of opinion as to whether or not they are subject to proprietary rights.

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

Contents

Preface xxi

Part I Concepts and Preliminaries

1 The Multilevel-Multiscale Approach 5

1.1 The Multilevel-Multiscale Concept 5

1.2 The Integer Number 6

1.3 Division of Integers 8

1.4 The Greatest Common Divisor 9

1.5 Multilevel Refinement 10

1.6 Example in Computer Science 10

1.7 Multilevel in Mathematical Logic 11

1.8 Multilevel in Language 11

1.9 Multilevel Programming 12

1.10 Object-Oriented Programming 12

1.11 Example in Data Structures 13

1.12 The Sorting Problem 13

1.13 Parallelism 13

1.14 Self-Similarity 14

1.15 The Wavelet Transform 14

1.16 Mathematical Induction and Recursion 14

1.17 The Tower Problem 15

1.18 The Parallel Product Algorithm 17

1.19 Multilevel in Statistics 18

1.20 Multilevel in Music 19

1.21 Exercises 19

2 Preliminaries 23

2.1 Preliminary Notation and Definitions 23

2.2 Application in Pivoting 31

2.3 Standard Lemmas about Symmetric Matrices 31

2.4 The Fourier Transform 38

2.5 Exercises 41

Part II Partial Differential Equations and Their Discretization

3	Finite Differences and Volumes	49
3.1	Elliptic PDEs	49
3.2	The Diffusion Equation	50
3.3	The Finite-Difference Discretization Method	51
3.4	Error Estimate	53
3.5	Finite Differences for the Poisson Equation	54
3.6	Error Estimate for Diffusion Problems	55
3.7	The Indefinite Helmholtz Equation	55
3.8	Adequate Discretization of the Helmholtz Equation	57
3.9	Adequate Discretization of Highly Anisotropic Equations	59
3.10	Oblique Anisotropy	60
3.11	Finite Differences for the Convection-Diffusion Equation	61
3.12	The Finite-Volume Discretization Method	63
3.13	Exercises	65
4	Finite Elements	67
4.1	The Finite-Element Discretization Method	67
4.2	The Weak Formulation	67
4.3	The Discrete Weak Formulation	69
4.4	Bilinear Finite Elements	71
4.5	Triangulation	72
4.6	Diagonal Dominance in the Isotropic Case	74
4.7	Diagonal Dominance in the Anisotropic Case	76
4.8	Locally Refined Meshes	77
4.9	The Refinement Step	78
4.10	Adaptive Mesh Refinement	80
4.11	Exercises	82

Part III The Numerical Solution of Large Sparse Linear Systems

5	Iterative Linear System Solvers	89
5.1	Iterative Sparse Linear System Solvers	89
5.2	Relaxation Methods	90
5.3	The Jacobi Relaxation Method	90
5.4	The Damped Jacobi Relaxation Method	91
5.5	The Block Jacobi Relaxation Method	91
5.6	The Gauss-Seidel Relaxation Method	91
5.7	The Block-Gauss-Seidel Relaxation Method	92
5.8	Reordering by Colors	92
5.9	Four-Color Reordering	95
5.10	Cache-Oriented Reordering	96
5.11	Symmetric Gauss-Seidel Relaxation	99
5.12	The Preconditioned Conjugate Gradient Method	100
5.13	Incomplete LU Factorization (ILU)	102

5.14	Parallelizable ILU Version	102
5.15	Nonsymmetric and Indefinite Problems	104
5.16	Numerical Comparison	105
5.17	The Normal Equations	106
5.18	Exercises	106
6	The Multigrid Iteration	109
6.1	The Two-Grid Method	109
6.2	Transfer and Coarse-Grid Operators	112
6.3	The Multigrid Method	112
6.4	Geometric Multigrid	114
6.5	Variational Multigrid	115
6.6	Domain Decomposition and Variational Multigrid	116
6.7	Domain Decomposition and Black-Box Multigrid	118
6.8	Domain Decomposition and Algebraic Multigrid	120
6.9	The Algebraic Multilevel Method	122
6.10	Algebraic Multigrid	124
6.11	Semicoarsening	126
6.12	Exercises	128
Part IV Multigrid for Structured Grids		
7	Automatic Multigrid	135
7.1	Properties of the AutoMUG Method	135
7.2	Cyclic Reduction	136
7.3	The Two-Dimensional Case	137
7.4	Definition of the AutoMUG Method	138
7.5	The AutoMUG(q) Version	141
7.6	Exercises	143
8	Applications in Image Processing	145
8.1	The Denoising Problem	145
8.2	The Denoising Algorithm for Grayscale Images	146
8.3	The Denoising Algorithm for Color Images	147
8.4	Numerical Examples	149
8.5	Exercises	152
9	Black-Box Multigrid	155
9.1	Definition of Black-Box Multigrid	155
9.2	Improvements in Diffusion Problems	156
9.3	Using the Right Hand Side	159
9.4	Improvement for Problems with Discontinuous Coefficients	160
9.5	Exercise	164

10	The Indefinite Helmholtz Equation	165
10.1	Multigrid for the Indefinite Helmholtz Equation	165
10.2	Improved Prolongation	166
10.3	Improved Black-Box Multigrid	167
10.4	Computational Two-Level Analysis	168
10.5	Multiple Coarse-Grid Corrections	171
10.6	The Size of the Coarsest Grid	174
10.7	Numerical Examples	175
10.8	Exercises	178
11	Matrix-Based Semicoarsening	183
11.1	The Semicoarsening Approach	183
11.2	Flow of Information in Elliptic PDEs	184
11.3	Multilevel Line Reordering	185
11.4	Block-ILU Factorization	187
11.5	The Domain-Decomposition Direct Solver	189
11.6	Reordered Block-ILU Factorization	192
11.7	Matrix-Based Semicoarsening	193
11.8	A Deblurring Problem	195
11.9	Exercises	197

Part V Multigrid for Semistructured Grids

12	Multigrid for Locally Refined Meshes	203
12.1	Multigrid and Hierarchical-Basis Linear System Solvers	203
12.2	The Two-Level Method	204
12.3	Matrix-Induced Inner Products and Norms	210
12.4	Properties of the Two-Level Method	211
12.5	Isotropic Diffusion Problems	214
12.6	Instability and Local Anisotropy	216
12.7	The Multilevel Method	216
12.8	Upper Bound for the Condition Number	218
12.9	Exercises	221
13	Application to Semistructured Grids	223
13.1	Semistructured Grids	223
13.2	The V-Cycle	224
13.3	The AFAC and AFACx Cycles	224
13.4	The Numerical Examples	226
13.5	Scaling the Coefficient Matrix	230
13.6	A Black-Box Multigrid Version	232
13.7	Exercises	233

Part VI Multigrid for Unstructured Grids

14	Domain Decomposition	239
14.1	The Domain-Decomposition Approach	239
14.2	The Domain-Decomposition Multigrid Method	240
14.3	Upper-Bound for the Condition Number	244
14.4	High-Order Finite Elements and Spectral Elements	245
14.5	Exercises	247
15	The Algebraic Multilevel Method	249
15.1	The Need for Algebraic Multilevel Methods	249
15.2	The Algebraic Multilevel Method	250
15.3	The Coarsening Procedure	251
15.4	The Transfer and Coarse-Level Matrices	252
15.5	The Relaxation Method	254
15.6	Properties of the Two-Level Method	255
15.7	Properties of the Multilevel Method	256
15.8	Upper-Bound for the Condition Number	256
15.9	The Approximate Schur Complement Method	258
15.10	Exercises	258
16	Applications	261
16.1	Highly Anisotropic Equations	261
16.2	Two-Step Jacobi Relaxation	262
16.3	The Maxwell Equations	263
16.4	The Convection-Diffusion Equation	264
16.5	ILU Relaxation	267
16.6	Towards Algebraic Semicoarsening	268
16.7	A Diffusion Problem in a Complicated Domain	268
16.8	Exercises	271
17	Semialgebraic Multilevel for Systems of PDEs	273
17.1	Semialgebraic Multilevel Methods	273
17.2	Standard Differential Operators	274
17.3	The Linear Elasticity Equations	275
17.4	The Weak Formulation	275
17.5	The Finite-Element Discretization	276
17.6	The Semialgebraic Multilevel Preconditioner	277
17.7	Preconditioner for the Stokes Equations	278
17.8	The Reduced Linear Elasticity Equations	280
17.9	Towards Problems with Constraints	281
17.10	Towards Semialgebraic Block Lumping	282
17.11	A Domain-Decomposition Two-Level Method	283
17.12	Exercises	285

Part VII Appendices

18 Time-Dependent Parabolic PDEs 293

18.1 Parabolic PDEs 293

18.2 The Parabolic Diffusion Equation 293

18.3 The Weak Formulation 294

18.4 The Semi-Implicit Time Discretization 294

18.5 The Finite-Element Discretization 295

18.6 Stability Analysis 296

18.7 Accuracy of the Numerical Scheme 299

18.8 The Algebraic Multilevel Preconditioner 299

19 Nonlinear Equations 301

19.1 Nonlinear PDEs 301

19.2 The Residual Equation 301

19.3 Defect Correction 302

19.4 Geometric Multigrid 303

19.5 The Newton Iteration 304

References 305

Index 313