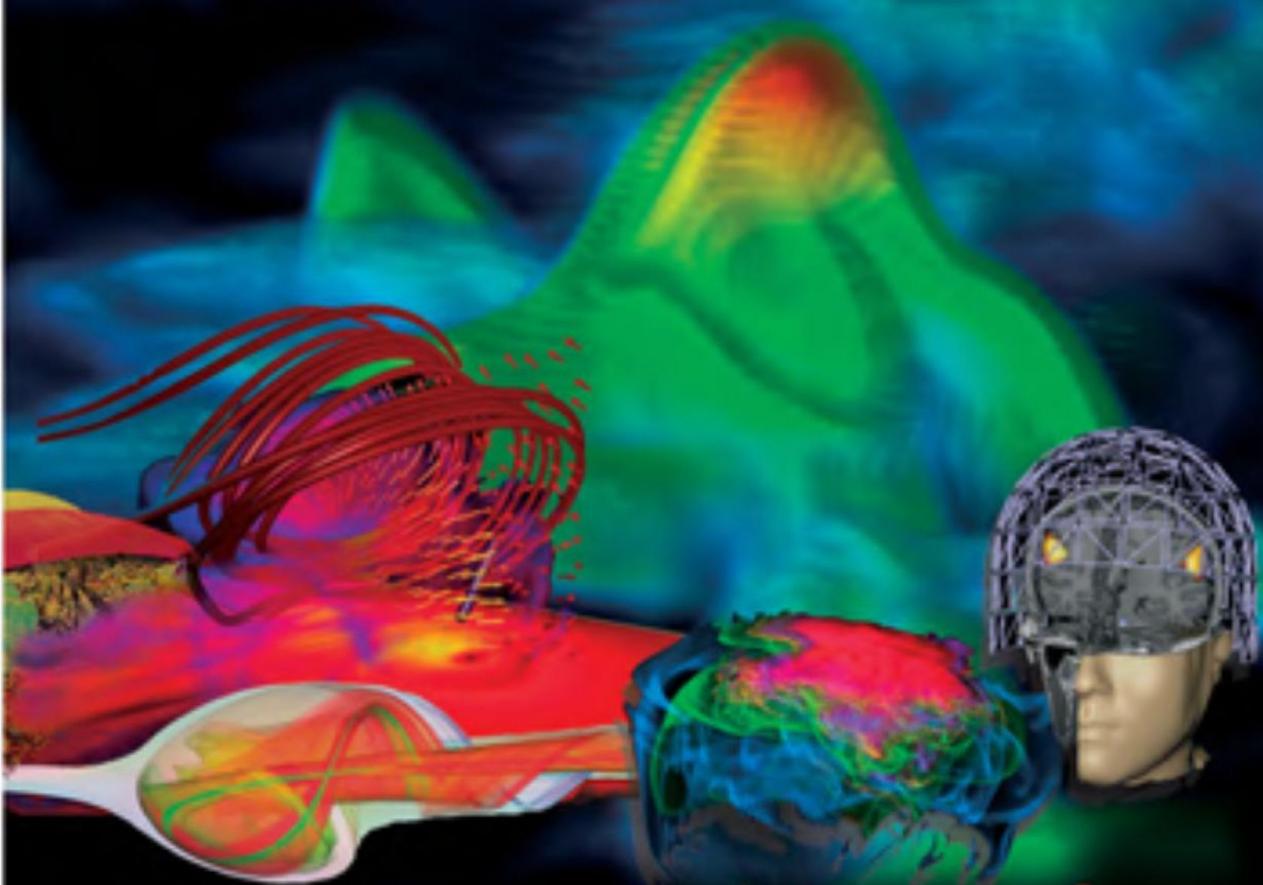


# Numerical Methods for Evolutionary Differential Equations



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COMPUTATIONAL SCIENCE & ENGINEERING

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