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Arithmetical Investigations

Representation Theory, Orthogonal Polynomials, and Quantum Interpolations

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