

Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains

Volume II

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