History of Modern Physics and Astronomy Volume 13

New Methods of Celestial Mechanics

Parts 1, 2, and 3
by Henri Poincaré

originally published as *Les Méthodes nouvelles de la Mécanique céleste*

CONTENTS
PART 1

PERIODIC AND ASYMPTOTIC SOLUTIONS

PAGE

Preface to the French Edition xxi
Introduction
by Daniel L. Goroff I1

CHAPTER 1
GENERALITIES AND THE JACOBI METHOD 1

Generalities 1
Examples of Canonical Equations 2
First Jacobi Theorem 6
Second Jacobi Theorem; Changes of Variables . . . 7
Remarkable Changes of Variables 8
Keplerian Motion 11
Special Case of the Three-Body Problem 13
Use of Keplerian Variables 15
General Case of the Three-Body Problem 17
General Problem of Dynamics 22
Reduction of the Canonical Equations 22
Reduction of the Three-Body Problem 26
Form of the Perturbative Function 29
Invariant Relations 33

CHAPTER 2
SERIES INTEGRATION 35

Definitions and Various Lemmas 35
Cauchy's Theorem 38
Extension of Cauchy's Theorem 43

Applications to the Three-Body Problem 46
Use of Trigonometric Series 48
Implicit Functions 52
Algebraic Singular Points 54
Elimination 55
Theorem of the Maxima 57
New Definitions 59

CHAPTER 3
PERIODIC SOLUTIONS 61

Case When Time Does Not
Enter Explicitly into the Equations 69
Application to the Three-Body Problem 75
First-Type Solutions 76
Hill's Researches Concerning the Moon 82
Application to the
General Problem of Dynamics 86
Case Where the Hessian is Zero 93
Direct Calculation of Series 95
Direct Demonstration of Convergence 102
Examination of an Important Exceptional Case . . 107
Solution of the Second Kind 112
Solution of the Third Kind 117
Applications of Periodic Solutions 123
Satellites of Jupiter 125
Periodic Solutions in the
Neighborhood of a Position of Equilibrium . . 126
Moons Without Quadrature 129

CHAPTER 4
CHARACTERISTIC EXPONENTS 131

Equations of Variation 131
Application to Lunar Theory 133
Equations of Variation of Dynamics 134

Application of the
Theory of Linear Substitutions 139
Definition of Characteristic Exponents 143
The Equation Which Defines These Exponents 144
Case Where Time Does Not Enter Explicitly 146
New Statement of the Theorem of Nos. 37 and 38 . . . 146
Case Where the
Equations Admit Uniform Integrals 150
Case of Equations of Dynamics 157
Changes of Variables 162
Development of Exponents:
Calculation of the First Terms 164
Application to the Three-Body Problem 178
Complete Calculation of
Characteristic Exponents 179
Degenerate Solutions 187

CHAPTER 5
NONEXISTENCE OF
UNIFORM INTEGRALS 191

Case Where the *B* Vanish 196
Case Where the Hessian is Zero 200
Application to the Three-Body Problem 205
Problems of Dynamics
Where There Exists a Uniform Integral 208
Nonholomorphic Integrals in μ 212
Discussion of Expressions (5.14a) 214

CHAPTER 6
APPROXIMATE DEVELOPMENT
OF THE PERTURBATIVE FUNCTION 220

Statement of the Problem 220
Digression on a Property
of the Perturbative Function 223
Principles of the Method of Darboux 228
Extensions to Functions of Several Variables 230

Investigation of Singular Points 234
Discussion 241
Discussion of the General Case 251
Application of the Method of Darboux 258
Application to Astronomy 266
Application to Demonstration of the
Nonexistence of Uniform Integrals 267

CHAPTER 7
ASYMPTOTIC SOLUTIONS 275

Convergence of Series 277
Asymptotic Solutions
of the Equations of Dynamics 282
Development of These Solutions in Powers of $\sqrt{\mu}$ . . 283
Divergence of the Series of No. 108 288
New Demonstration of the Proposition of No. 108 . . 290
Transformation of Equations 297
Reduction to the Canonical Form 303
Form of Functions V_i 305
Fundamental Lemma 307
Analogy of the Series
of No. 108 with that of Stirling 311

Poincaré's Footnotes 315
Russian Endnotes E1
Index xxv

CONTENTS
PART 2

APPROXIMATIONS BY SERIES

PAGE

Preface to Part 2 of the French Edition xxi
Review of Notations xxiii

CHAPTER 8
FORMAL CALCULUS 317

Various Meanings of the Term "Convergence" 317
Series Analogous to the Stirling Series 318
Calculation of the Series 320

CHAPTER 9
METHODS OF NEWCOMB AND LINDSTEDT 328

Historical Background 328
Discussion of the Method 330
Various Forms of Series 335
Direct Calculation of the Series 338
Comparison with the Newcomb Method 344

CHAPTER 10
APPLICATION TO THE STUDY
OF SECULAR VARIATIONS 347

Discussion of the Question 347
New Change of Variables 348
Application of the Method of Chap. 9 352

CHAPTER 11
APPLICATION TO THE THREE-BODY PROBLEM 355

Difficulty of the Problem 355
Extension of the Method of
Chap. 9 to Certain Singular Cases 356
Application to the Three-Body Problem 362
Change of Variables 363
Case of Plane Orbits 365
Study of a Particular Integral 371
Form of the Expansions 372
General Case of the Three-Body Problem 374

CHAPTER 12
APPLICATION TO ORBITS 378

Discussion of the Difficulties Involved 378
Solution of the Difficulty 384

CHAPTER 13
DIVERGENCE OF THE LINDSTEDT SERIES 395

Discussion of Series (13.3) 395
Discussion of Series (13.2) 400
Comparison with the Old Methods 404

CHAPTER 14
DIRECT CALCULATION OF THE SERIES 410

Generalities 410
Application to the Three-Body Problem 423
Diverse Properties 432
Noteworthy Particular Cases 444
Conclusions 449

CHAPTER 15
OTHER METHODS OF DIRECT CALCULATION 450

Problem of No. 125 450
Another Example 453
Problem of No. 134 460
Three-Body Problem 468

CHAPTER 16
GYLDÉN METHODS 490

Reduction of the Equations 498
Intermediate Orbit 507
Absolute Orbit 508

CHAPTER 17
CASE OF LINEAR EQUATIONS 511

Study of the Gyldén Equation 511
Jacobi Method 528
Gyldén's Method 531
Bruns Method 534
Lindstedt Method 535
Hill's Method 539
Application of the Hadamard Theorem 545
Miscellaneous Remarks 552
Extension of the Preceding Results 554

CHAPTER 18
CASE OF NONLINEAR EQUATIONS 557

Inhomogeneous Equations 557
Equation of the Evection 560
Variational Equation 575
Summary 581
Generalization of Periodic Solutions 582

CHAPTER 19
BOHLIN METHODS 586

Delaunay Method 586
Bohlin Method 609
Case of Libration 617
Limiting Case 628
Correlation with the Series No. 125 642
Divergence of the Series 646

CHAPTER 20
BOHLIN SERIES 652

Case of Libration 656
Limiting Case 660
Comparison with the Series of No. 127 672

CHAPTER 21
EXTENSION OF THE BOHLIN METHOD 679

Extension of the Problem of No. 134 679
Extension of the Three-Body Problem 687
Discussion of the Series 690
Second Method 694
Case of Libration 698
Divergence of the Series 700

Poincaré's Footnotes 722
Russian Endnotes E7
Index xxv

CONTENTS
PART 3

INTEGRAL INVARIANTS AND ASYMPTOTIC PROPERTIES OF CERTAIN SOLUTIONS

PAGE

CHAPTER 22
INTEGRAL INVARIANTS 723

Steady Motion of a Fluid 723
Definition of Integral Invariants 725
Relationships Between
the Invariants and the Integrals 728
Relative Invariants 729
Relationship Between the
Invariants and the Variational Equation 735
Transformation of the Invariants 738
Other Relationships
Between the Invariants and the Integrals 745
Change in Variables 748
General Remarks 750

CHAPTER 23
FORMATION OF INVARIANTS 758

Use of the Last Multiplier 758
Equations of Dynamics 760
Integral Invariants and Characteristic Exponents . . . 765
Use of Kepler Variables 777
Remarks on the Invariant Given in No. 256 781
Case of the Reduced Problem 783

CHAPTER 24
USE OF INTEGRAL INVARIANTS 784

Test Procedures 784
Relationship to a Jacobi Theorem 791
Application to the Two-Body Problem 794
Application to Asymptotic Solutions 798

CHAPTER 25
INTEGRAL INVARIANTS AND ASYMPTOTIC SOLUTIONS 801

Return to the Method of Bohlin 801
Relationship with Integral Invariants 822
Another Discussion Method 827
Quadratic Invariants 836
Case of the Restricted Problem 840

CHAPTER 26
POISSON STABILITY 847

Different Definitions of Stability 847
Motion of a Liquid 848
Probabilities 856
Extension of the Preceding Results 859
Application to the Restricted Problem 861
Application to the Three-Body Problem 868

CHAPTER 27
THEORY OF CONSEQUENTS 877

Invariant Curves 880
Extension of the Preceding Results 887
Application to Equations of Dynamics 890
Application to the Restricted Problem 895

CHAPTER 28
PERIODIC SOLUTIONS OF THE SECOND KIND

PERIODIC SOLUTIONS OF THE SECOND KIND . . . 900

Case in Which Time Does Not Enter Explicitly . . . 904
Application to the Equations of Dynamics . . . 910
Solutions of the Second Kind for Equations of Dynamics . . . 922
Theorems Considering the Maxima . . . 926
Existence of Solutions of the Second Kind . . . 934
Remarks . . . 937
Special Cases . . . 939

CHAPTER 29
DIFFERENT FORMS OF THE PRINCIPLE OF LEAST ACTION

DIFFERENT FORMS OF THE PRINCIPLE OF LEAST ACTION . . . 942

Kinetic Focus . . . 952
Maupertuis Focus . . . 958
Application to Periodic Solutions . . . 960
Case of Stable Solutions . . . 961
Unstable Solutions . . . 963

CHAPTER 30
FORMATION OF SOLUTIONS OF THE SECOND KIND

FORMATION OF SOLUTIONS OF THE SECOND KIND . . . 981

Effective Formulation of the Solutions . . . 982
Discussion . . . 995
Discussion of Particular Cases . . . 1005
Application to Equations of No. 13 . . . 1006

CHAPTER 31
PROPERTIES OF SOLUTIONS OF THE SECOND KIND

PROPERTIES OF SOLUTIONS OF THE SECOND KIND . . . 1012

Solutions of the Second Kind and the Principle of Least Action . . . 1012
Stability and Instability . . . 1022
Application to the Orbits of Darwin . . . 1029

CHAPTER 32
PERIODIC SOLUTIONS
OF THE SECOND KIND 1037

CHAPTER 33
DOUBLY ASYMPTOTIC SOLUTIONS 1045

Different Methods of Geometric Representation . . . 1045
Homoclinic Solutions 1055
Heteroclinic Solutions 1061
Comparison with No. 225 1063
Examples of Heteroclinic Solutions 1066

Russian Endnotes E19
Index xxi