

Terry J. Lyons
Michael Caruana · Thierry Lévy

Differential Equations Driven by Rough Paths

École d'Été de Probabilités
de Saint-Flour XXXIV - 2004

 Springer

Contents

1	Differential Equations Driven by Moderately Irregular Signals	1
1.1	Signals with Bounded Variation	1
1.1.1	The General Setting of Controlled Differential Equations	1
1.1.2	The Theorems of Picard-Lindelöf and Cauchy-Peano	2
1.2	Paths with Finite p -Variation	4
1.2.1	Definitions	4
1.2.2	Controls	6
1.2.3	Approximation Results	7
1.3	The Young Integral	10
1.4	Differential Equations Driven by Signals with Finite p -Variation, with $p < 2$	13
1.4.1	Peano's Theorem	13
1.4.2	Lipschitz Functions	15
1.4.3	Picard's Theorem	20
1.5	What Goes Wrong with $p \geq 2$	21
1.5.1	No Continuous Extension of the Strichartz Integral is Rich Enough to Handle Brownian Paths	21
1.5.2	The Area Enclosed by a Path is Not a Continuous Functional in Two-Variation	22
2	The Signature of a Path	25
2.1	Iterated Integrals and Linear Equations	25
2.2	The Signature of a Path	28
2.2.1	Formal Series of Tensors	29
2.2.2	The Signature of a Path	30
2.2.3	Functions on the Range of the Signature	31
2.2.4	Lie Elements, Logarithm and Exponential	36
2.2.5	Truncated Signature and Free Nilpotent Groups	37
2.2.6	The Signature of Paths with Bounded Variation	39

3	Rough Paths	41
3.1	Multiplicative Functionals	41
3.1.1	Definition of Multiplicative Functionals	41
3.1.2	Extension of Multiplicative Functionals	45
3.1.3	Continuity of the Extension Map	51
3.2	Spaces of Rough Paths	52
3.2.1	Rough Paths and the p -Variation Topology	52
3.2.2	Geometric Rough Paths	53
3.3	The Brownian Rough Path	55
3.3.1	The Itô Multiplicative Functional	55
3.3.2	The Stratonovich Multiplicative Functional	56
3.3.3	New Noise Sources	58
4	Integration Along Rough Paths	63
4.1	Almost-Multiplicativity	63
4.1.1	Almost-Additivity	63
4.1.2	Almost Rough Paths	65
4.2	Linear Differential Equations Driven by Rough Paths	69
4.3	Integration of a One-form Along a Rough Path	73
4.3.1	Construction of an Almost Rough Path	73
4.3.2	Definition of the Integral	75
5	Differential Equations Driven by Rough Paths	81
5.1	Linear Images of Geometric Rough Paths	81
5.2	Solution of a Differential Equation Driven by a Rough Path	82
5.3	The Universal Limit Theorem	83
5.4	Linear Images and Comparison of Rough Paths	84
5.5	Three Picard Iterations	86
5.6	The Main Scaling Result	88
5.7	Uniform Control of the Picard Iterations	89
5.8	Proof of the Main Theorem	91
	References	95
	Index	101
	List of Participants	103
	List of Short Lectures	107