

David J. Green

# Gröbner Bases and the Computation of Group Cohomology



Springer

# Table of Contents

<b>Introduction .....</b>	<b>1</b>
<hr/>	
<b>Part I Constructing minimal resolutions</b>	
<hr/>	
<b>1 Bases for finite-dimensional algebras and modules .....</b>	<b>13</b>
1.1 Finite-dimensional algebras .....	14
1.2 Free right modules.....	16
1.3 Implementation .....	17
1.4 The matrix of a general element.....	17
1.5 The Jennings ordering .....	18
<b>2 The Buchberger Algorithm for modules .....</b>	<b>21</b>
2.1 The Diamond Lemma for modules .....	22
2.1.1 Reduction systems .....	22
2.1.2 Ambiguities and the Diamond Lemma .....	23
2.2 The Buchberger Algorithm .....	26
2.3 Implementation .....	29
2.3.1 The algorithm <code>Incorp</code> .....	29
2.3.2 The Buchberger Algorithm .....	31
<b>3 Constructing minimal resolutions .....</b>	<b>33</b>
3.1 The kernel of a homomorphism .....	33
3.2 Minimal generating sets .....	35
3.2.1 Heavily reduction systems .....	36
3.2.2 Algorithms for heavily reduction systems .....	37
3.3 Implementation .....	39
3.3.1 The variants of the algorithm <code>Incorp</code> .....	40
3.3.2 The variants of the Buchberger Algorithm .....	42
3.4 Computing preimages .....	45

---

**Part II Cohomology ring structure**

<b>4</b>	<b>Gröbner bases for graded commutative algebras</b>	49
4.1	The structure of $\Theta$ -algebras	50
4.2	Gröbner bases for right ideals	55
4.2.1	Gröbner bases and the Division Algorithm	55
4.2.2	The Buchberger Algorithm	56
4.3	The kernel of an algebra homomorphism	59
4.4	Intersections and Annihilators: Gröbner bases for modules	62
<b>5</b>	<b>The visible ring structure</b>	67
5.1	Basics	67
5.2	Practical considerations	68
5.2.1	Lifts of cocycles	68
5.2.2	Gröbner bases and the visible ring structure	69
5.3	Monomial ordering and generator choice	72
5.3.1	Nilpotent generators	72
5.3.2	Regular generators	74
5.3.3	In summary	74
5.4	Calculating products in batches	75
5.5	Restriction to subgroups	79
<b>6</b>	<b>The completeness of the presentation</b>	81
6.1	The Koszul complex	81
6.2	Carlson's completeness criterion	83
6.3	Duflo regular sequences	84
6.4	Groups of small rank: Koszul complex and Poincaré series	86
6.5	Identifying subgroups	89

---

**Part III Experimental results**

<b>7</b>	<b>Experimental results</b>	93
7.1	Cohomology rings of small $p$ -groups	93
7.1.1	The groups of order 81	95
7.1.2	The groups of order 625	96
7.1.3	Two essential classes with nonzero product	96
7.1.4	A 3-group with Cohen–Macaulay defect 2	98
7.2	Resolutions for larger $p$ -groups	99
7.3	The period of a periodic module	100

<b>A Sample cohomology calculations . . . . .</b>	101
A.1 The cyclic group of order 2 . . . . .	101
A.2 The cyclic group of order 4 . . . . .	101
A.3 The Klein 4-group . . . . .	101
A.4 The dihedral group of order 8 . . . . .	102
A.5 The quaternion group of order 8 . . . . .	103
A.6 The Sylow 2-subgroup of $U_3(4)$ . . . . .	105
A.7 The Sylow 3-subgroup of $A_9$ . . . . .	123
A.8 Small Group No. 16 of order 243 . . . . .	126
<b>Epilogue . . . . .</b>	131
<b>References . . . . .</b>	133
<b>Index . . . . .</b>	137