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# Mathematical Analysis and Numerical Methods for Science and Technology

Volume 6  
Evolution Problems II

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