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# Mathematical Analysis and Numerical Methods for Science and Technology

Volume 2

Functional and Variational Methods

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# Table of Contents

## Chapter III. Functional Transformations

|                                                                                                |    |
|------------------------------------------------------------------------------------------------|----|
| Introduction . . . . .                                                                         | 1  |
| <i>Part A. Some Transformations Useful in Applications</i> . . . . .                           | 4  |
| §1. Fourier Series and Dirichlet's Problem . . . . .                                           | 4  |
| 1. Fourier Series . . . . .                                                                    | 4  |
| 1.1. Convergence in $L^2(\mathbb{T})$ . . . . .                                                | 5  |
| 1.2. Pointwise Convergence on $\mathbb{T}$ . . . . .                                           | 5  |
| 2. Distributions on $\mathbb{T}$ and Periodic Distributions . . . . .                          | 7  |
| 2.1. Comparison of $\mathcal{D}'(\mathbb{T})$ with the Distributions on $\mathbb{R}$ . . . . . | 7  |
| 2.2. Principal Properties of $\mathcal{D}'(\mathbb{T})$ . . . . .                              | 9  |
| 3. Fourier Series of Distributions . . . . .                                                   | 10 |
| 4. Fourier Series and Fourier Transforms . . . . .                                             | 14 |
| 5. Convergence in the Sense of Césaro . . . . .                                                | 15 |
| 6. Solution of Dirichlet's Problem with the Help of Fourier Series . . . . .                   | 17 |
| 6.1. Dirichlet's Problem in a Disk . . . . .                                                   | 17 |
| 6.2. Dirichlet's Problem in a Rectangle . . . . .                                              | 20 |
| §2. The Mellin Transform . . . . .                                                             | 24 |
| 1. Generalities . . . . .                                                                      | 24 |
| 2. Definition of the Mellin Transform . . . . .                                                | 26 |
| 3. Properties of the Mellin Transform . . . . .                                                | 28 |
| 4. Inverse Mellin Transform . . . . .                                                          | 30 |
| 5. Applications of the Mellin Transform . . . . .                                              | 32 |
| 6. Table of Some Mellin Transforms . . . . .                                                   | 40 |
| §3. The Hankel Transform . . . . .                                                             | 40 |
| 1. Generalities . . . . .                                                                      | 40 |
| 2. Introduction to Bessel Functions . . . . .                                                  | 42 |
| 3. Definition of the Hankel Transform . . . . .                                                | 47 |
| 4. The Inversion Formula . . . . .                                                             | 48 |
| 5. Properties of the Hankel Transform . . . . .                                                | 50 |
| 6. Application of the Hankel Transform to Partial Differential Equations . . . . .             | 53 |

|                                                                                                                         |    |
|-------------------------------------------------------------------------------------------------------------------------|----|
| 6.1. Dirichlet's Problem for Laplace's Equation in $\mathbb{R}^2$ ,<br>The Case of Axial Symmetry . . . . .             | 53 |
| 6.2. Boundary Value Problem for the Biharmonic Equation<br>in $\mathbb{R}^2$ , with Axial Symmetry . . . . .            | 55 |
| 7. Table of Some Hankel Transforms . . . . .                                                                            | 57 |
| Review of Chapter III A . . . . .                                                                                       | 57 |
| <i>Part B. Discrete Fourier Transforms and Fast Fourier Transforms</i> . . . . .                                        | 59 |
| §1. Introduction . . . . .                                                                                              | 59 |
| §2. Acceleration of the Product of a Matrix by a Vector . . . . .                                                       | 62 |
| §3. The Fast Fourier Transform of Cooley and Tukey . . . . .                                                            | 64 |
| §4. The Fast Fourier Transform of Good-Winograd . . . . .                                                               | 66 |
| §5. Reduction of the Number of Multiplications . . . . .                                                                | 69 |
| 1. Relation Between the Discrete Fourier Transform<br>and the Problem of Cyclic Convolution . . . . .                   | 69 |
| 2. Complexity of the Product of Two Polynomials . . . . .                                                               | 71 |
| 3. Application to the Cyclic Convolution of Order 2 . . . . .                                                           | 72 |
| 4. Application to the Cyclic Convolution of Order 3 . . . . .                                                           | 73 |
| 5. Application to the Cyclic Convolution of Order 6 . . . . .                                                           | 75 |
| §6. Fast Fourier Transform in Two Dimensions . . . . .                                                                  | 77 |
| §7. Some Applications of the Fast Fourier Transform . . . . .                                                           | 78 |
| 1. Solution of Boundary Value Problems . . . . .                                                                        | 78 |
| 2. Regularisation and Smoothing of Functions . . . . .                                                                  | 81 |
| 3. Practical Calculation of the Fourier Transform of a Signal . . . . .                                                 | 83 |
| 4. Determination of the Spectrum of Certain Finite Difference<br>Operators and Fast Solvers for the Laplacian . . . . . | 84 |
| Review of Chapter III B . . . . .                                                                                       | 91 |

#### Chapter IV. Sobolev Spaces

|                                                                                                                                  |     |
|----------------------------------------------------------------------------------------------------------------------------------|-----|
| Introduction . . . . .                                                                                                           | 92  |
| §1. Spaces $H^1(\Omega)$ , $H^m(\Omega)$ . . . . .                                                                               | 92  |
| §2. The Space $H^p(\mathbb{R}^n)$ . . . . .                                                                                      | 96  |
| 1. Definition and First Properties . . . . .                                                                                     | 96  |
| 2. The Topological Dual of $H^p(\mathbb{R}^n)$ . . . . .                                                                         | 98  |
| 3. The Equation $(-\Delta + k^2)u = f$ in $\mathbb{R}^n$ , $k \in \mathbb{R} \setminus \{0\}$ . . . . .                          | 100 |
| §3. Sobolev's Embedding Theorem . . . . .                                                                                        | 100 |
| §4. Density and Trace Theorems for the Spaces $H^m(\Omega)$ ,<br>( $m \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ ) . . . . . | 102 |

|                                                                                                                                |     |
|--------------------------------------------------------------------------------------------------------------------------------|-----|
| 1. A Density Theorem . . . . .                                                                                                 | 102 |
| 2. A Trace Theorem for $H^1(\mathbb{R}_+^n)$ . . . . .                                                                         | 107 |
| 3. Traces of the Spaces $H^m(\mathbb{R}_+^n)$ and $H^m(\Omega)$ . . . . .                                                      | 113 |
| 4. Properties of $m$ -Extension . . . . .                                                                                      | 114 |
| §5. The Spaces $H^{-m}(\Omega)$ for all $m \in \mathbb{N}$ . . . . .                                                           | 120 |
| §6. Compactness . . . . .                                                                                                      | 123 |
| §7. Some Inequalities in Sobolev Spaces . . . . .                                                                              | 125 |
| 1. Poincaré's Inequality for $H_0^1(\Omega)$ (resp. $H_0^m(\Omega)$ ) . . . . .                                                | 125 |
| 2. Poincaré's Inequality for $H^1(\Omega)$ . . . . .                                                                           | 127 |
| 3. Convexity Inequalities for $H^m(\Omega)$ . . . . .                                                                          | 133 |
| §8. Supplementary Remarks . . . . .                                                                                            | 138 |
| 1. Sobolev Spaces $W^{m,p}(\Omega)$ . . . . .                                                                                  | 138 |
| 1.1. Definitions . . . . .                                                                                                     | 138 |
| 1.2. Sobolev Injections . . . . .                                                                                              | 139 |
| 1.3. Trace Theorems for the Spaces $W^{m,p}(\Omega)$ . . . . .                                                                 | 140 |
| 2. Sobolev Spaces with Weights . . . . .                                                                                       | 141 |
| 2.1. Unbounded Open Sets . . . . .                                                                                             | 141 |
| 2.2. Polygonal Open Sets . . . . .                                                                                             | 141 |
| Review of Chapter IV . . . . .                                                                                                 | 142 |
| Appendix: The Spaces $H^s(\Gamma)$ with $\Gamma$ the "Regular" Boundary<br>of an Open Set $\Omega$ in $\mathbb{R}^n$ . . . . . | 143 |

## Chapter V. Linear Differential Operators

|                                                                                                     |     |
|-----------------------------------------------------------------------------------------------------|-----|
| Introduction . . . . .                                                                              | 148 |
| §1. Generalities on Linear Differential Operators . . . . .                                         | 149 |
| 1. Characterisation of Linear Differential Operators . . . . .                                      | 149 |
| 2. Various Definitions . . . . .                                                                    | 152 |
| 2.1. Leibniz's Formula . . . . .                                                                    | 152 |
| 2.2. Transpose of a Linear Differential Operator . . . . .                                          | 153 |
| 2.3. Order of a Linear Differential Operator . . . . .                                              | 154 |
| 3. Linear Differential Operator on a Manifold . . . . .                                             | 155 |
| 4. Characteristics . . . . .                                                                        | 157 |
| 4.1. Concept of Characteristics . . . . .                                                           | 157 |
| 4.2. Bicharacteristics . . . . .                                                                    | 159 |
| 5. Operators with Analytic Coefficients.<br>Theorems of Cauchy-Kowalewsky and of Holmgren . . . . . | 163 |
| §2. Linear Differential Operators with Constant Coefficients . . . . .                              | 170 |
| 1. Study of a l.d.o. with Constant Coefficients<br>by the Fourier Transform . . . . .               | 171 |

|                                                                                                             |     |
|-------------------------------------------------------------------------------------------------------------|-----|
| 1.1. Existence of a Solution of $Pu=f$ in the Space of Tempered Distributions . . . . .                     | 171 |
| 1.2. Example 1: The Laplacian . . . . .                                                                     | 173 |
| 1.3. Elliptic and Strongly Elliptic Operators . . . . .                                                     | 174 |
| 1.4. Hypo-Elliptic and Semi-Elliptic Operators . . . . .                                                    | 177 |
| 1.5. Examples . . . . .                                                                                     | 179 |
| 1.6. Reduction of Operators of Order 2 in a Homogeneous, Isotropic "Medium" . . . . .                       | 181 |
| 2. Elementary Solutions of a l.d.o. with Constant Coefficients . . . . .                                    | 182 |
| 2.1. Introduction . . . . .                                                                                 | 182 |
| 2.2. Elementary Solutions in $\mathcal{S}'$ Examples . . . . .                                              | 184 |
| 2.3. Elementary Solution with Support in a Salient Closed Convex Cone: Hyperbolic Operator . . . . .        | 189 |
| 3. Characterisation of Hyperbolic Operators . . . . .                                                       | 191 |
| 3.1. Characteristics of a l.d.o. with Constant Coefficients . . . . .                                       | 191 |
| 3.2. Algebraic Characterisation of Hyperbolic Operators . . . . .                                           | 194 |
| 3.3. Hyperbolic Operators of Order 2 . . . . .                                                              | 198 |
| 4. Parabolic Operators . . . . .                                                                            | 202 |
| §3. Cauchy Problem for Differential Operators with Constant Coefficients . . . . .                          | 204 |
| 1. Cauchy Problem and the Elementary Solution in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^+)$ . . . . . | 205 |
| 2. Propagation in Hyperbolic Cauchy Problems . . . . .                                                      | 209 |
| 3. Choice of a Functional Space: Well-Posed Cauchy Problem . . . . .                                        | 214 |
| 4. Well-Posed Cauchy Problem in $\mathcal{S}'$ . . . . .                                                    | 217 |
| 5. Parabolic and Weakly Parabolic Operators . . . . .                                                       | 221 |
| 6. Study of the Particular Case $P = \partial/\partial t + P_0$ . . . . .                                   | 223 |
| 6.1. Analysis of One-Dimensional Case . . . . .                                                             | 223 |
| 6.2. Case in which $P_0$ is Strongly Elliptic . . . . .                                                     | 224 |
| 6.3. Schrödinger Operator . . . . .                                                                         | 225 |
| 7. Well-Posed Cauchy Problem in $\mathcal{S}'$ : Hyperbolic Operators . . . . .                             | 226 |
| §4. Local Regularity of Solutions <sup>a</sup> . . . . .                                                    | 230 |
| 1. Characterisation of Hypo-Ellipticity . . . . .                                                           | 230 |
| 1.1. Necessary Condition for Hypo-Ellipticity . . . . .                                                     | 230 |
| 1.2. Algebraic Transformation of the Necessary Condition for Hypo-Ellipticity . . . . .                     | 232 |
| 1.3. The Principal Result . . . . .                                                                         | 233 |
| 2. Analyticity of Solutions . . . . .                                                                       | 234 |
| 2.1. Statement of Results . . . . .                                                                         | 234 |
| 2.2. Estimates of Analyticity . . . . .                                                                     | 237 |
| 2.3. Generalisation: Gevrey Classes . . . . .                                                               | 240 |
| 3. Comparison of Operators . . . . .                                                                        | 241 |
| 4. Local Regularity for Operators with Variable Coefficients and of Constant Force . . . . .                | 245 |
| 5. Construction of an Elementary Solution . . . . .                                                         | 247 |

|                                                                       |     |
|-----------------------------------------------------------------------|-----|
| §5. The Maximum Principle*                                            | 250 |
| 1. Prerequisites                                                      | 250 |
| 2. Parabolic Maximum Principle and Dissipativity                      | 252 |
| 3. Characterisation of Operators $P$ Satisfying<br>Maximum Principles | 259 |
| 3.1. The Weak Maximum Principle                                       | 259 |
| 3.2. The Comparison Principle                                         | 261 |
| 3.3. The Strong Maximum Principle                                     | 263 |
| 3.4. The Principle of the Strong Parabolic Maximum                    | 265 |
| Review of Chapter V                                                   | 268 |

## Chapter VI. Operators in Banach Spaces and in Hilbert Spaces

|                                                                                         |     |
|-----------------------------------------------------------------------------------------|-----|
| Introduction                                                                            | 269 |
| §1. Review of Functional Analysis: Banach and Hilbert Spaces                            | 270 |
| 1. Locally Convex Topological Vector Spaces, Normed Spaces<br>and Banach Spaces         | 270 |
| 2. Linear Operators                                                                     | 274 |
| 3. Duality                                                                              | 281 |
| 4. The Hahn-Banach Theorem and its Applications                                         | 282 |
| 4.1. Problems of Approximation                                                          | 282 |
| 4.2. Problems of Existence                                                              | 283 |
| 4.3. Problems of Separation of Convex Sets                                              | 285 |
| 5. Bidual, Reflexivity, Weak Convergence, Weak Compactness                              | 285 |
| 5.1. Bidual                                                                             | 285 |
| 5.2. Reflexivity                                                                        | 286 |
| 5.3. Weak Convergence                                                                   | 287 |
| 5.4. Weak Compactness                                                                   | 289 |
| 5.5. Weak-Star Convergence                                                              | 290 |
| 6. Hilbert Spaces                                                                       | 291 |
| 6.1. Definitions                                                                        | 291 |
| 6.2. Projection on a Closed Convex Set                                                  | 295 |
| 6.3. Orthonormal Bases                                                                  | 299 |
| 6.4. The Riesz Representation Theorem, Reflexivity                                      | 302 |
| 7. Ideas About Functions of a Real or Complex Variable<br>with Values in a Banach Space | 304 |
| 7.1. Weak Topology                                                                      | 304 |
| 7.2. Weak Differentiability                                                             | 304 |
| 7.3. Weak Holomorphy                                                                    | 305 |
| §2. Linear Operators in Banach Spaces                                                   | 305 |
| 1. Generalities on Linear Operators                                                     | 305 |
| 1.1. Domain, Kernel and Image of a Linear Operator                                      | 305 |

|                                                                                                                     |     |
|---------------------------------------------------------------------------------------------------------------------|-----|
| 1.2. Nullity and Deficiency Indices . . . . .                                                                       | 307 |
| 1.3. Basic Properties of Linear Operators . . . . .                                                                 | 307 |
| 2. Spaces of Bounded Operators . . . . .                                                                            | 310 |
| 2.1. Introduction . . . . .                                                                                         | 310 |
| 2.2. Various Concepts of Convergence of Operators . . . . .                                                         | 313 |
| 2.3. Composition and Inverse of Bounded Operators . . . . .                                                         | 316 |
| 2.4. Transpose of a Bounded Operator . . . . .                                                                      | 322 |
| 2.5. Some Classes of Bounded Operators . . . . .                                                                    | 325 |
| 2.6. Some Ideas on Functions of a Real or Complex Variable<br>with Operator Values; Families of Operators . . . . . | 332 |
| 3. Closed Operators . . . . .                                                                                       | 334 |
| 3.1. Definition and Examples . . . . .                                                                              | 334 |
| 3.2. Basic Properties . . . . .                                                                                     | 336 |
| 3.3. The Set $\mathcal{F}(X, Y)$ of Closed Operators from $X$ into $Y$ . . . . .                                    | 339 |
| 3.4. Transpose of a Closed Operator . . . . .                                                                       | 342 |
| 3.5. Operators with Closed Image . . . . .                                                                          | 346 |
| §3. Linear Operators in Hilbert Spaces . . . . .                                                                    | 348 |
| 1. Bounded Operators in Hilbert Spaces . . . . .                                                                    | 351 |
| 1.1. Adjoint Sesquilinear Form . . . . .                                                                            | 353 |
| 1.2. Hermitian Operators . . . . .                                                                                  | 353 |
| 1.3. Orthogonal Projectors . . . . .                                                                                | 354 |
| 1.4. Isometries and Unitary Operators . . . . .                                                                     | 355 |
| 1.5. Hilbert-Schmidt Operators . . . . .                                                                            | 357 |
| 2. Unbounded Operators in Hilbert Spaces . . . . .                                                                  | 361 |
| 2.1. Adjoint of an Unbounded Operator . . . . .                                                                     | 361 |
| 2.2. Symmetric Operators . . . . .                                                                                  | 361 |
| 2.3. The Cayley Transform . . . . .                                                                                 | 362 |
| 2.4. Normal Operators . . . . .                                                                                     | 366 |
| 2.5. Sesquilinear Forms and Unbounded Operators . . . . .                                                           | 367 |
| Review of Chapter VI . . . . .                                                                                      | 374 |

## Chapter VII. Linear Variational Problems. Regularity

|                                                                                                         |     |
|---------------------------------------------------------------------------------------------------------|-----|
| Introduction . . . . .                                                                                  | 375 |
| §1. Elliptic Variational Theory . . . . .                                                               | 375 |
| 1. The Lax-Milgram Theorem . . . . .                                                                    | 376 |
| 2. First Examples . . . . .                                                                             | 378 |
| 2.1. Example 1. Dirichlet Problem . . . . .                                                             | 379 |
| 2.2. Example 2. Neumann Problem . . . . .                                                               | 380 |
| 3. Extensions in the Case in which $V$ and $H$ are Spaces<br>of Distributions or of Functions . . . . . | 383 |
| 4. Sesquilinear Forms Associated with Elliptic Operators<br>of Order Two . . . . .                      | 384 |

|                                                                                                                     |     |
|---------------------------------------------------------------------------------------------------------------------|-----|
| 5. Sesquilinear Forms Associated with Elliptic Operators of Order $2m$ . . . . .                                    | 387 |
| 6. Miscellaneous Remarks . . . . .                                                                                  | 389 |
| 7. Application to the Solution of General Elliptic Problems (of Dirichlet Type) . . . . .                           | 391 |
| §2. Examples of Second Order Elliptic Problems . . . . .                                                            | 393 |
| 1. Generalities . . . . .                                                                                           | 393 |
| 2. Examples of Variational Problems . . . . .                                                                       | 394 |
| 2.1. Mixed Problem . . . . .                                                                                        | 394 |
| 2.2. Non-Local Boundary Conditions . . . . .                                                                        | 397 |
| 3. Problems Relative to Integro-Differential Forms on $\Omega + \Gamma$ . . . . .                                   | 398 |
| 3.1. Problem of the Oblique Derivative . . . . .                                                                    | 398 |
| 3.2. Robin's Problem . . . . .                                                                                      | 400 |
| 4. Transmission Problem . . . . .                                                                                   | 400 |
| 5. Miscellaneous Remarks . . . . .                                                                                  | 405 |
| 6. Application: Stationary Multigroup Equation for the Diffusion of Neutrons . . . . .                              | 407 |
| 7. Application: Static Problems of Elasticity . . . . .                                                             | 411 |
| 7.1. Introduction . . . . .                                                                                         | 411 |
| 7.2. Variational Formulation . . . . .                                                                              | 412 |
| 7.3. Korn's Inequality . . . . .                                                                                    | 414 |
| 7.4. Application to Problem (2.39) . . . . .                                                                        | 418 |
| 7.5. Inhomogeneous Problem . . . . .                                                                                | 418 |
| 8. Static Problems of the Flexure of Plates . . . . .                                                               | 420 |
| §3. Regularity of the Solutions of Variational Problems . . . . .                                                   | 425 |
| 1. Introduction . . . . .                                                                                           | 425 |
| 2. Interior Regularity . . . . .                                                                                    | 426 |
| 3. Global Regularity of the Solutions of Dirichlet and Neumann Problems for Elliptic Operators of Order 2 . . . . . | 433 |
| 4. Miscellaneous Results on Global Regularity . . . . .                                                             | 437 |
| 5. Green's Functions . . . . .                                                                                      | 441 |
| 5.1. Case of the Laplacian in a Bounded Open Set $\Omega$ with Dirichlet Condition . . . . .                        | 441 |
| 5.2. Some Other Particular Examples . . . . .                                                                       | 445 |
| 5.3. Green's Functions in a More General Setting . . . . .                                                          | 451 |
| Review of Chapter VII . . . . .                                                                                     | 456 |

## Appendix. "Distributions"

|                                                                |     |
|----------------------------------------------------------------|-----|
| §1. Definition and Basic Properties of Distributions . . . . . | 457 |
| 1. The Space $\mathcal{D}(\Omega)$ . . . . .                   | 457 |
| 1.1. Definition . . . . .                                      | 457 |

|                                                                                                                               |     |
|-------------------------------------------------------------------------------------------------------------------------------|-----|
| 1.2. Elementary Properties of the Convolution Product of Two Functions . . . . .                                              | 458 |
| 1.3. A Procedure for the Construction of Functions of $\mathcal{D}(\Omega)$ . . . . .                                         | 459 |
| 1.4. The Notion of Convergence in $\mathcal{D}(\Omega)$ . . . . .                                                             | 460 |
| 1.5. Some Inclusion and Density Properties . . . . .                                                                          | 461 |
| 2. The Space $\mathcal{D}'(\Omega)$ of Distributions on $\Omega$ . . . . .                                                    | 463 |
| 2.1. Definition of Distributions and the Concept of Convergence in $\mathcal{D}'(\Omega)$ . . . . .                           | 463 |
| 2.2. First Examples of Distributions: Measures on $\Omega$ . . . . .                                                          | 464 |
| 2.3. Differentiation of Distributions. Examples . . . . .                                                                     | 467 |
| 2.4. Support of a Distribution. Distributions with Compact Support . . . . .                                                  | 474 |
| 3. Some Elementary Operations on Distributions . . . . .                                                                      | 476 |
| 3.1. Product by a Function of Class $\mathcal{C}^\infty$ . . . . .                                                            | 476 |
| 3.2. Primitives of a Distribution on a Interval of $\mathbb{R}$ . . . . .                                                     | 477 |
| 3.3. Tensor Product of Two Distributions . . . . .                                                                            | 480 |
| 3.4. Direct Image and Inverse Image of a Function and of a Distribution by a Function of Class $\mathcal{C}^\infty$ . . . . . | 481 |
| 4. Some Examples . . . . .                                                                                                    | 482 |
| 4.1. Primitives of the Dirac Measure . . . . .                                                                                | 482 |
| 4.2. A Division Problem (Case $n = 1$ ) . . . . .                                                                             | 483 |
| 4.3. Derivative of a Function of $\mathbb{R}^n$ Discontinuous on a Surface . . . . .                                          | 484 |
| 4.4. Distributions Defined by Inverse Image from Distributions on the Real Line . . . . .                                     | 485 |
| §2. Convolution of Distributions . . . . .                                                                                    | 492 |
| 1. Convolution of a Distribution on $\mathbb{R}^n$ and a Function of $\mathcal{D}(\mathbb{R}^n)$ . . . . .                    | 492 |
| 2. Convolution of Two Distributions of Which One (at Least) is with Compact Support . . . . .                                 | 494 |
| 3. Distributions with Convolutional Supports . . . . .                                                                        | 496 |
| 4. Convolution Algebras . . . . .                                                                                             | 497 |
| §3. Fourier Transforms . . . . .                                                                                              | 500 |
| 1. Fourier Transform of $L^1$ -Functions . . . . .                                                                            | 500 |
| 2. The Space $\mathcal{S}'(\mathbb{R}^n)$ . . . . .                                                                           | 502 |
| 3. Fourier Transform in $L^2$ . . . . .                                                                                       | 506 |
| 4. Fourier Transforms of Tempered Distributions . . . . .                                                                     | 506 |
| 5. Fourier Transform of Distributions with Compact Support . . . . .                                                          | 509 |
| 6. Examples of the Calculation of Fourier Transforms . . . . .                                                                | 510 |
| 7. Partial Fourier Transform . . . . .                                                                                        | 513 |
| 8. Fourier Transform and Automorphisms of $\mathbb{R}^n$ : Homogeneous Distributions . . . . .                                | 518 |
| 8.1. Fourier Transform and Automorphisms of $\mathbb{R}^n$ . . . . .                                                          | 518 |
| 8.2. Homogeneous Distributions . . . . .                                                                                      | 519 |

|                                                                                                                            |     |
|----------------------------------------------------------------------------------------------------------------------------|-----|
| 9. Fourier Transform and Convolution. Spaces $\mathcal{C}_M(\mathbb{R}^n)$<br>and $\mathcal{C}'_C(\mathbb{R}^n)$ . . . . . | 520 |
| 9.1. The Space $\mathcal{C}_M(\mathbb{R}^n)$ ( $= \mathcal{C}_M$ ) . . . . .                                               | 520 |
| 9.2. The Space $\mathcal{C}'_C$ . . . . .                                                                                  | 521 |
| 10. Fourier Transform of Tempered Measures . . . . .                                                                       | 523 |
| 11. Distribution of Positive Type. Bochner's Theorem . . . . .                                                             | 525 |
| 11.1. Functions of Positive Type . . . . .                                                                                 | 525 |
| 11.2. Distributions of Positive Type . . . . .                                                                             | 525 |
| 12. Schwartz's Theorem of Kernels . . . . .                                                                                | 527 |
| 13. Some Distributions and Their Fourier Transforms . . . . .                                                              | 532 |
| Bibliography . . . . .                                                                                                     | 533 |
| Table of Notations . . . . .                                                                                               | 538 |
| Index . . . . .                                                                                                            | 551 |
| Contents of Volumes 1, 3-6 . . . . .                                                                                       | 557 |