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# Problems and Theorems in Linear Algebra



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$$|A + B| \geq |A| \left( 1 + \sum_{k=1}^{n-1} \frac{|B_k|}{|A_k|} \right) + |B| \left( 1 + \sum_{k=1}^{n-1} \frac{|A_k|}{|B_k|} \right).$$

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**40.3. Theorem.** Let  $A, B$  be matrices such that  $\text{ad}_A^s X = 0$  implies  $\text{ad}_X^s B = 0$  for some  $s > 0$ . Then  $B = g(A)$  for a polynomial  $g$ .

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$$(x_1^2 + \dots + x_n^2)(y_1^2 + \dots + y_n^2) = (z_1^2 + \dots + z_n^2),$$

where  $z_i(x, y)$  is a bilinear function, holds if and only if  $m \leq \rho(n)$ .

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