

# **COMBINATORIAL ALGORITHMS**

*Generation, Enumeration,  
and Search*

**Donald L. Kreher**

*Department of Mathematical Sciences  
Michigan Technological University*

**Douglas R. Stinson**

*Department of Combinatorics and Optimization  
University of Waterloo*



CRC Press

Boca Raton London New York Washington, D.C.

---

## Contents

---

<b>1 Structures and Algorithms</b>	1
1.1 What are combinatorial algorithms?	1
1.2 What are combinatorial structures?	2
1.2.1 Sets and lists	2
1.2.2 Graphs	4
1.2.3 Set systems	5
1.3 What are combinatorial problems?	7
1.4 O-Notation	9
1.5 Analysis of algorithms	10
1.5.1 Average-case complexity	12
1.6 Complexity classes	13
1.6.1 Reductions between problems	16
1.7 Data structures	17
1.7.1 Data structures for sets	17
1.7.2 Data structures for lists	22
1.7.3 Data structures for graphs and set systems	22
1.8 Algorithm design techniques	23
1.8.1 Greedy algorithms	23
1.8.2 Dynamic programming	24
1.8.3 Divide-and-conquer	25
1.9 Notes	26
Exercises	27
<b>2 Generating Elementary Combinatorial Objects</b>	31
2.1 Combinatorial generation	31
2.2 Subsets	32
2.2.1 Lexicographic ordering	32
2.2.2 Gray codes	35
2.3 k-Element subsets	43
2.3.1 Lexicographic ordering	43
2.3.2 Co-lex ordering	45
2.3.3 Minimal change ordering	48

2.4	Permutations . . . . .	52
2.4.1	Lexicographic ordering . . . . .	52
2.4.2	Minimal change ordering . . . . .	57
2.5	Notes . . . . .	64
	Exercises . . . . .	64
<b>3</b>	<b>More Topics in Combinatorial Generation</b>	<b>67</b>
3.1	Integer partitions . . . . .	67
3.1.1	Lexicographic ordering . . . . .	74
3.2	Set partitions, Bell and Stirling numbers . . . . .	78
3.2.1	Restricted growth functions . . . . .	81
3.2.2	Stirling numbers of the first kind . . . . .	87
3.3	Labeled trees . . . . .	91
3.4	Catalan families . . . . .	95
3.4.1	Ranking and unranking . . . . .	98
3.4.2	Other Catalan families . . . . .	101
3.5	Notes . . . . .	103
	Exercises . . . . .	103
<b>4</b>	<b>Backtracking Algorithms</b>	<b>105</b>
4.1	Introduction . . . . .	105
4.2	A general backtrack algorithm . . . . .	107
4.3	Generating all cliques . . . . .	109
4.3.1	Average-case analysis . . . . .	112
4.4	Estimating the size of a backtrack tree . . . . .	115
4.5	Exact cover . . . . .	118
4.6	Bounding functions . . . . .	122
4.6.1	The knapsack problem . . . . .	123
4.6.2	The traveling salesman problem . . . . .	127
4.6.3	The maximum clique problem . . . . .	135
4.7	Branch and bound . . . . .	141
4.8	Notes . . . . .	144
	Exercises . . . . .	145
<b>5</b>	<b>Heuristic Search</b>	<b>151</b>
5.1	Introduction to heuristic algorithms . . . . .	151
5.1.1	Uniform graph partition . . . . .	155
5.2	Design strategies for heuristic algorithms . . . . .	156
5.2.1	Hill-climbing . . . . .	157
5.2.2	Simulated annealing . . . . .	158
5.2.3	Tabu search . . . . .	160
5.2.4	Genetic algorithms . . . . .	161
5.3	A steepest ascent algorithm for uniform graph partition	165
5.4	A hill-climbing algorithm for Steiner triple systems .	167

## CONTENTS

5.4.1	Implementation details . . . . .	170
5.4.2	Computational results . . . . .	174
5.5	Two heuristic algorithms for the knapsack problem . . . . .	175
5.5.1	A simulated annealing algorithm . . . . .	175
5.5.2	A tabu search algorithm . . . . .	178
5.6	A genetic algorithm for the traveling salesman problem . . . . .	181
5.7	Notes . . . . .	186
	Exercises . . . . .	189
<b>6</b>	<b>Groups and Symmetry</b>	<b>191</b>
6.1	Groups . . . . .	191
6.2	Permutation groups . . . . .	195
6.2.1	Basic algorithms . . . . .	199
6.2.2	How to store a group . . . . .	201
6.2.3	Schreier-Sims algorithm . . . . .	203
6.2.4	Changing the base . . . . .	211
6.3	Orbits of subsets . . . . .	213
6.3.1	Burnside's lemma . . . . .	214
6.3.2	Computing orbit representatives . . . . .	217
6.4	Coset representatives . . . . .	223
6.5	Orbits of $k$ -tuples . . . . .	224
6.6	Generating objects having automorphisms . . . . .	226
6.6.1	Incidence matrices . . . . .	227
6.7	Notes . . . . .	232
	Exercises . . . . .	232
<b>7</b>	<b>Computing Isomorphism</b>	<b>237</b>
7.1	Introduction . . . . .	237
7.2	Invariants . . . . .	238
7.3	Computing certificates . . . . .	245
7.3.1	Trees . . . . .	245
7.3.2	Graphs . . . . .	253
7.3.3	Pruning with automorphisms . . . . .	264
7.4	Isomorphism of other structures . . . . .	272
7.4.1	Using known automorphisms . . . . .	272
7.4.2	Set systems . . . . .	272
7.5	Notes . . . . .	275
	Exercises . . . . .	275
<b>8</b>	<b>Basis Reduction</b>	<b>277</b>
8.1	Introduction . . . . .	277
8.2	Theoretical development . . . . .	281
8.3	A reduced basis algorithm . . . . .	291
8.4	Solving systems of integer equations . . . . .	294

8.5 The Merkle-Hellman knapsack system . . . . .	300
8.6 Notes . . . . .	306
Exercises . . . . .	307
<b>Bibliography</b>	<b>311</b>
<b>Algorithm Index</b>	<b>319</b>
<b>Problem Index</b>	<b>323</b>
<b>Index</b>	<b>325</b>