

Martino Bardi
Italo Capuzzo-Dolcetta

Optimal Control
and Viscosity Solutions
of Hamilton-Jacobi-Bellman
Equations

Birkhäuser
Boston • Basel • Berlin

Contents

Preface	xI
Basic notations	xv
Chapter I. Outline of the main ideas on a model problem	1
1. The infinite horizon discounted regulator	1
2. The Dynamic Programming Principle	2
3. The Hamilton-Jacobi-Bellman equation in the viscosity sense	3
4. Comparison, uniqueness and stability of viscosity solutions	6
5. Synthesis of optimal controls and verification theorems	10
6. Pontryagin Maximum Principle as a necessary and sufficient condition of optimality	13
7. Discrete time Dynamic Programming and convergence of approximations	17
8. The viscosity approximation and stochastic control	20
9. Bibliographical notes	21
Chapter II. Continuous viscosity solutions of Hamilton-Jacobi equations	25
1. Definitions and basic properties	25
2. Some calculus and further properties of viscosity solutions	34
3. Some comparison and uniqueness results	50
4. Lipschitz continuity and semiconcavity	61
4.1. Lipschitz continuity	62
4.2. Semiconcavity	65
5. Special results for convex Hamiltonians	77
5.1. Semiconcave generalized solutions and bilateral supersolutions	77
5.2. Differentiability of solutions	80
5.3. A comparison theorem	82
5.4. Solutions in the extended sense	84

5.5. Differential inequalities in the viscosity sense	86
5.6. Monotonicity of value functions along trajectories	90
6. Bibliographical notes	95
Chapter III. Optimal control problems with continuous value functions: unrestricted state space	97
1. The controlled dynamical system	97
2. The infinite horizon problem	99
2.1. Dynamic Programming and the Hamilton-Jacobi-Bellman equation	99
2.2. Some simple applications: verification theorems, relaxation, stability	110
2.3. Backward Dynamic Programming, sub- and superoptimality principles, bilateral solutions	119
2.4. Generalized directional derivatives and equivalent notions of solution	125
2.5. Necessary and sufficient conditions of optimality, minimum principles, and multivalued optimal feedbacks	133
3. The finite horizon problem	147
3.1. The HJB equation	147
3.2. Local comparison and unbounded value functions	156
3.3. Equivalent notions of solution	160
3.4. Necessary and sufficient conditions of optimality and the Pontryagin Maximum Principle	170
4. Problems whose HJB equation is a variational or quasivariational inequality	183
4.1. The monotone control problem	184
4.2. Optimal stopping	193
4.3. Impulse control	200
4.4. Optimal switching	210
5. Appendix: Some results on ordinary differential equations	218
6. Bibliographical notes	223
Chapter IV. Optimal control problems with continuous value functions: restricted state space	227
1. Small-time controllability and minimal time functions	227
2. HJB equations and boundary value problems for the minimal time function: basic theory	239
3. Problems with exit times and non-zero terminal cost	247
3.1. Compatible terminal cost and continuity of the value function	248
3.2. The HJB equation and a superoptimality principle	254
4. Free boundaries and local comparison results for undiscounted problems with exit times	261
5. Problems with state constraints	271
6. Bibliographical notes	282

Chapter V. Discontinuous viscosity solutions and applications	285
1. Semicontinuous sub- and supersolutions, weak limits, and stability	286
2. Non-continuous solutions	295
2.1. Definitions, basic properties, and examples	295
2.2. Existence of solutions by Perron's method	302
3. Envelope solutions of Dirichlet problems	308
3.1. Existence and uniqueness of ϵ -solutions	309
3.2. Time-optimal problems lacking controllability	313
4. Boundary conditions in the viscosity sense	316
4.1. Motivations and basic properties	317
4.2. Comparison results and applications to exit-time problems and stability	326
4.3. Uniqueness and complete solution for time-optimal control	333
5. Bilateral supersolutions	342
5.1. Problems with exit times and general targets	342
5.2. Finite horizon problems with constraints on the endpoint of the trajectories	348
6. Bibliographical notes	357
Chapter VI. Approximation and perturbation problems	359
1. Semidiscrete approximation and ϵ -optimal feedbacks	359
1.1. Approximation of the value function and construction of optimal controls	360
1.2. A first result on the rate of convergence	369
1.3. Improving the rate of convergence	372
2. Regular perturbations	376
3. Stochastic control with small noise and vanishing viscosity	383
4. Appendix: Dynamic Programming for Discrete Time Systems	388
5. Bibliographical notes	395
Chapter VII. Asymptotic problems	397
1. Ergodic problems	397
1.1. Vanishing discount in the state constrained problem	397
1.2. Vanishing discount in the unconstrained case: optimal stopping	402
2. Vanishing switching costs	411
3. Penalization	414
3.1. Penalization of stopping costs	414
3.2. Penalization of state constraints	417
4. Singular perturbation problems	420
4.1. The infinite horizon problem for systems with fast components	420
4.2. Asymptotics for the monotone control problem	426
5. Bibliographical notes	429

Chapter VIII. Differential Games	431
1. Dynamic Programming for lower and upper values	433
2. Existence of a value, relaxation, verification theorems	442
3. Comparison with other information patterns and other notions of value	448
3.1. Feedback strategies	448
3.2. Approximation by discrete time games	457
4. Bibliographical notes	468
Appendix A. Numerical Solution of Dynamic Programming Equations by Maurizio Falcone	471
1. The infinite horizon problem	472
1.1. The Dynamic Programming equation	473
1.2. Synthesis of feedback controls	478
1.3. Numerical tests	481
2. Problems with state constraints	484
3. Minimum time problems and pursuit-evasion games	488
3.1. Time-optimal control	488
3.2. Pursuit-evasion games	494
3.3. Numerical tests	496
4. Some hints for the construction of the algorithms	499
5. Bibliographical notes	502
Appendix B. Nonlinear \mathcal{H}_∞ control by Pierpaolo Soravia	505
1. Definitions	506
2. Linear systems	510
3. \mathcal{H}_∞ control and differential games	512
4. Dynamic Programming equation	515
5. On the partial information problem	521
6. Solving the problem	527
7. Exercises	530
8. Bibliographical notes	531
Bibliography	533
Index	565