

W. Hackbusch

Elliptic Differential Equations

Theory and Numerical Treatment

Translated from the German
by Regine Fadiman and Patrick D. F. Ion

With 40 Figures



Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

Table of Contents

Foreword	V
Table of Contents	VII
Notation	XIII
 1 Partial Differential Equations and Their Classification Into Types	 1
1.1 Examples	1
1.2 Classification of Second-Order Equations into Types	4
1.3 Type Classification for Systems of First Order	6
1.4 Characteristic Properties of the Different Types	7
 2 The Potential Equation	 12
2.1 Posing the Problem	12
2.2 Singularity Function	14
2.3 The Mean Value Property and Maximum Principle	17
2.4 Continuous Dependence on the Boundary Data	23
 3 The Poisson Equation	 27
3.1 Posing the Problem	27
3.2 Representation of the Solution by the Green Function	28
3.3 The Green Function for the Ball	34
3.4 The Neumann Boundary Value Problem	35
3.5 The Integral Equation Method	36
 4 Difference Methods for the Poisson Equation	 38
4.1 Introduction: The One-Dimensional Case	38
4.2 The Five-Point Formula	40
4.3 M -matrices, Matrix Norms, Positive Definite Matrices	44
4.4 Properties of the Matrix L_h	53
4.5 Convergence	59

4.6	Discretisations of Higher Order	62
4.7	The Discretisation of the Neumann Boundary Value Problem	65
4.7.1	One-sided Difference for $\partial u / \partial n$	65
4.7.2	Symmetric Difference for $\partial u / \partial n$	70
4.7.3	Symmetric Difference for $\partial u / \partial n$ on an Offset Grid	71
4.7.4	Proof of the Stability Theorem 7	72
4.8	Discretisation in an Arbitrary Domain	78
4.8.1	Shortley-Weller Approximation	78
4.8.2	Interpolation at Points near the Boundary	83
5	General Boundary Value Problems	85
5.1	Dirichlet Boundary Value Problems for Linear Differential Equations	85
5.1.1	Posing the Problem	85
5.1.2	Maximum Principle	86
5.1.3	Uniqueness of the Solution and Continuous Dependence	87
5.1.4	Difference Methods for the General Differential Equation of Second Order	90
5.1.5	Green's Function	95
5.2	General Boundary Conditions	95
5.2.1	Formulating the Boundary Value Problem	95
5.2.2	Difference Methods for General Boundary Conditions	98
5.3	Boundary Problems of Higher Order	103
5.3.1	The Biharmonic Differential Equation	103
5.3.2	General Linear Differential Equations of Order $2m$	104
5.3.3	Discretisation of the Biharmonic Differential Equation	105
6	Tools from Functional Analysis	110
6.1	Banach Spaces and Hilbert Spaces	110
6.1.1	Normed Spaces	110
6.1.2	Operators	111
6.1.3	Banach Spaces	112
6.1.4	Hilbert Spaces	114
6.2	Sobolev Spaces	115
6.2.1	$L^2(\Omega)$	115
6.2.2	$H^k(\Omega)$ and $H_0^k(\Omega)$	116
6.2.3	Fourier Transformation and $H^k(\mathbb{R}^n)$	119

6.2.4	$H^s(\Omega)$ for Real $s \geq 0$	122
6.2.5	Trace and Extension Theorems	123
6.3	Dual Spaces	130
6.3.1	Dual Space of a Normed Space	130
6.3.2	Adjoint Operators	132
6.3.3	Scales of Hilbert Spaces	133
6.4	Compact Operators	135
6.5	Bilinear Forms	137
7	Variational Formulation	144
7.1	Historical Remarks	144
7.2	Equations with Homogeneous Dirichlet Boundary Conditions	145
7.3	Inhomogeneous Dirichlet Boundary Conditions	150
7.4	Natural Boundary Conditions	152
8	The Method of Finite Elements	161
8.1	The Ritz-Galerkin Method	161
8.2	Error Estimates	167
8.3	Finite Elements	171
8.3.1	Introduction: Linear Elements for $\Omega = (a, b)$	171
8.3.2	Linear Elements for $\Omega \subset \mathbb{R}^2$	174
8.3.3	Bilinear Elements for $\Omega \subset \mathbb{R}^2$	178
8.3.4	Quadratic Elements for $\Omega \subset \mathbb{R}^2$	180
8.3.5	Elements for $\Omega \subset \mathbb{R}^3$	182
8.3.6	Handling of Side Conditions	182
8.4	Error Estimates for Finite Element Methods	185
8.4.1	H^1 -Estimates for Linear Elements	185
8.4.2	L^2 and H^s Estimates for Linear Elements	190
8.5	Generalisations	193
8.5.1	Error Estimates for Other Elements	193
8.5.2	Finite Elements for Equations of Higher Order	194
8.5.2.1	Introduction: The One-Dimensional Biharmonic Equation	194
8.5.2.2	The Two-Dimensional Case	195
8.5.2.3	Estimating Errors	196
8.6	Finite Elements for Non-Polygonal Regions	196
8.7	Additional Remarks	199
8.7.1	Non-Conformal Elements	199

	8.7.2 The Trefftz Method	200
	8.7.3 Finite-Element Methods for Singular Solutions	201
	8.7.4 Adaptive Triangulation	201
	8.7.5 Hierarchical Bases	202
	8.7.6 Superconvergence	202
	8.8 Properties of the Stiffness Matrix	203
9	Regularity	208
9.1	Solutions of the Boundary Value Problem in $H^s(\Omega)$, $s > m$	208
9.1.1	The Regularity Problem	208
9.1.2	Regularity Theorems for $\Omega = \mathbb{R}^n$	210
9.1.3	Regularity Theorems for $\Omega = \mathbb{R}_+^n$	215
9.1.4	Regularity Theorems for General $\Omega \subset \mathbb{R}^n$	219
9.1.5	Regularity for Convex Domains and Domains with Corners	223
9.1.6	Regularity in the Interior	226
9.2	Regularity Properties of Difference Equations	226
9.2.1	Discrete H^1 -Regularity	226
9.2.2	Consistency	232
9.2.3	Optimal Error Estimates	238
9.2.4	H_h^2 -Regularity	240
10	Special Differential Equations	244
10.1	Differential Equations with Discontinuous Coefficients	244
10.1.1	Formulation	244
10.1.2	Discretisation	246
10.2	A Singular Perturbation Problem	247
10.2.1	The Convection-Diffusion Equation	247
10.2.2	Stable Difference Schemes	249
10.2.3	Finite Elements	251
11	Eigenvalue Problems	253
11.1	Formulation of Eigenvalue Problems	253
11.2	Finite Element Discretisation	254
11.2.1	Discretisation	254
11.2.2	Qualitative Convergence Results	256
11.2.3	Quantitative Convergence Results	260
11.2.4	Complementary Problems	264
11.3	Discretisation by Difference Methods	267

12	Stokes Equations	275
12.1	Systems of Elliptic Differential Equations	275
12.2	Variational Formulation	278
12.2.1	Weak Formulation of the Stokes Equations	278
12.2.2	Saddlepoint Problems	279
12.2.3	Existence and Uniqueness of the Solution of a Saddlepoint Problem	282
12.2.4	Solvability and Regularity of the Stokes Problem	285
12.2.5	A V_0 -elliptic Variational Formulation of the Stokes Problem	289
12.3	Mixed Finite-Element Method for the Stokes Problem	290
12.3.1	Finite-Element Discretisation of a Saddlepoint Problem	290
12.3.2	Stability Conditions	291
12.3.3	Stable Finite-Element Spaces for the Stokes Problem	293
12.3.3.1	Stability Criterion	293
12.3.3.2	Finite-Element Discretisations with the Bubble Function	294
12.3.3.3	Stable Discretisations with Linear Elements in V_h	296
12.3.3.4	Error Estimates	297
	Bibliography	300
	Index	307