IAS/PARK CITYMATHEMATICS SERIES

Volume 2

Nonlinear Partial Differential Equations in Differential Geometry

Robert Hardt Michael Wolf Editors



American Mathematical Society Institute for Advanced Study

Contents

Preface	xi
Introduction	1
Luis Caffarelli, A Priori Estimates and the Geometry of the Monge-	
Ampère Equation	.5
Introduction	7
Part 1. Interior A Priori Estimates for Solutions of Fully Non-linear Equa-	
tions	9
Statement of Main Theorems	10
Preliminary Tools	13
Harnack Inequality for Functions in $S^*(f)$	19
$W^{2,p}$ Estimates	22
Hölder Estimates	28
Part 2. Geometric Properties of the Monge Ampère Equation	.33
A Localization Property	34
Part 3. A Priori Estimates of Solutions to Monge Ampère Equations	41
The Correct Invariant Measure	41
Part 4. Interior $W^{2,p}$ Estimates for Solutions of the Monge Ampère Equation	47
Preliminary Results	48
Approximation Lemmas	.50
Tangent Paraboloids in Measure and Level Surface Estimates	53
References	63
Sun-Yung Alice Chang, The Moser-Trudinger Inequality and	
Applications to Some Problems in Conformal Geometry	65
Introduction	67
Lecture 1. Some Background Material	69
Preliminaries	69
Rayleigh Quotients	71
Weyl's Asymptotic Formula	73
References	74
Lecture 2. Ray-Singer-Polyakov Formula on Compact Surfaces	75
Heat Kernel	75
Asymptotic Behavior of the Trace of the Heat Kernel	76

CONTENTS

Ray-Singer-Polyakov Log Determinant Formula on a Compact Surface	
without Boundary	78
References	.81
Lecture 3. Moser-Onofri Inequality and Applications	83
Onofri's Inequality	83
Compactness of Isospectral Families	87
References	.88
Lecture 4. Existence of Extremal Functions for Moser Inequality	89
Proof of Theorem 2, for $n=2$	90
Proof of Lemma 1	92
References	94
Lecture 5. Beckner-Adams Inequalities and Extremal Log-determinants	95
Conformally covariant operators	95
Paneitz's Operator	.99
References	100
Lecture 6. Isospectral Compactness on 3-manifolds and Relation to the Yam-	
abe Problem	101
Condition (*)	104
Proof of Theorem 2'	106
References	109
Lecture 7. Prescribing Curvature Function on S^n	111
The Variation Functional F_K	113
Bounds for $S[w]$	115
Some Sufficient Conditions on K or R	121
References	124
Richard Schoen, The Effect of Curvature on the Behavior of	
Harmonic Functions and Mappings	127
Introduction	129
Lecture 1. Gradient Estimate and Comparison Theorems	133
Hessian Comparison Theorem	134
Laplacian Comparison Theorem	136
Proof of (*)	137
Lecture 2. Gradient Estimate Proof and Corollaries	141
Continuation of Proof	141
Harnack Inequality	144
A Liouville Corollary	145
Lecture 3. Harmonic Functions on Negatively Curved Manifolds	147
Dirichlet Problem	148
Related Results and Questions	149
Lecture 4. Harmonic Mapping into Singular Spaces	151
Motivation	151
Nonpositively Curved Metric Spaces	152

CONTENTS

A Two-dimensional Result	154
Lecture 5. Energy Convexity of Maps to an NPC Metric Space	157
Geodesic Homotopy	157
Monotonicity	160
Lecture 6. The Order Function	163
Interior Gradient Bound	164
Existence of a Homotopy Minimizer	166
Lecture 7. Approximation and Smoothness Results for Harmonic Maps	167
Intrinsic Homogeneity	167
Homogeneous Minimizers	168
Intrinsic Differentiability	171
Lecture 8. Order 1 Points and Partial Regularity	173
Proof of Singular Set Estimate	174
Lecture 9. Rigidity Results via Harmonic Maps	179
Superrigidity	179
A Vanishing Theorem	181
References	182
Leon Simon, Singularities of Geometric Variational Problems	185
Lecture 1. Basic Introductory Material	187
Definition of Energy Minimizing Map	187
Definition of Regular and Singular Set	188
The Variational Equations	188
The Monotonicity Formula	190
The Regularity Theorem	192
Corollaries of the Regularity Theorem	192
A Further Remark on Upper Semicontinuity of the Density	196
Lecture 2. Tangent Maps	197
Definition of Tangent Map	197
Properties of Homogeneous Degree Zero Minimizers	198
Further Properties of Sing u	200
Lecture 3. The Top-Dimensional Part of Sing u	205
Homogeneous Degree Zero φ with dim $S(\varphi) = n - 3$.205
The Geometric Picture Near Points of Sing, u	208
Consequences of Uniqueness of Tangent Maps	.210
Lecture 4. Recent Results Concerning Sing u	213
Statement of Main Known Results	213
Preliminary Remarks on the Method of Proof: "Blowing Up"	214
L^2 Estimates	216
Special Solutions of the Linearized Equation	219
Brief Sketch of the Proof of the Results of 4.1	220
References	222

viii CONTENTS

Leon Simon, Proof of the Basic Regularity Theorem for Harmonic	
Maps	225
Lecture 1. Analytic Preliminaries	227
Hölder Continuous Functions	228
Functions with L^2 Gradient	230
Harmonic Functions	231
Harmonic Approximation Lemma	232
Lecture 2. A General Regularity Lemma	235
Statement of Main Regularity Lemma and Remarks	235
Proof of the Regularity Lemma	236
Lecture 3. The Reverse Poincaré Inequality	239
A Lemma of Luckhaus, and Some Corollaries	239
Proof of the Reverse Poincaré Inequality	243
Lecture 4. Completion of the Regularity Proof	247
A Further Property of Functions with L^2 Gradient	247
Proof of Luckhaus' Lemma	248
Proof of $C^{1,\alpha}$ and higher regularity	253
References	256
Michael Struwe, Geometric Evolution Problems	257
Introduction	259
Part 1. The Evolution of Harmonic Maps	261
Harmonic Maps	261
Bochner Identity	263
Homotopy and Dirichlet Problems	265
The Eells-Sampson Result	267
Finite Time Blow-up	269
Global Existence and Uniqueness of Partially Regular Weak Solutions for $m=2$	272
m = 2 Applications	279
Existence of Global, Partially Regular Weak Solutions for $m \geq 3$	283
The Monotonicity Formula	285
Convergence of Penalized Solutions	288
Nonuniqueness	290
Development of Singularities	292
Singularities of First and Second Kind	293
Part 2. The Evolution of Hypersurfaces by Mean Curvature	295
Mean Curvature Flow	295
Compact Surfaces	296
Entire Graphs	298
Generalized Motion by Mean Curvature	300
Uniqueness, Comparison Principles, Global Existence	302
Monotonicity Formula	304
Consequences of the Monotonicity Estimate	307
Singularities	300

CONTENTS	i

Part 3. Harmonic Maps of Minkowsky Space	311
The Cauchy Problem for Harmonic Maps	311
Local Existence	313
Global Existence	318
Finite-time Blow-up	320
Self-similar Equivariant Solutions	320
Global Existence and Regularity for Equivariant Harmonic Maps for	
m = 2	324
Equivariant Harmonic Maps with Convex Range	331
Deferences	222