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Dynamical Systems I

Ordinary Differential Equations
and Smooth Dynamical Systems

With 25 Figures



Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo

I. Ordinary Differential Equations

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Translated from the Russian
by E. R. Dawson and D. O'Shea

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Preface

This survey is devoted mainly to the local theory of ordinary differential equations. It does not include bifurcation theory, which will be dealt with in a separate article. The averaging method is dealt with in the survey "Mathematical aspects of classical and celestial mechanics" by V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt (volume 3 of this series).

We do not touch on the spectral theory of differential operators with a single independent variable (see, for example, [28]); as regards objectives and methods, this is more closely related to functional analysis. Our survey also does not include the theory of integral transforms and their application to linear differential equations. The asymptotic theory of differential equations is dealt with in M. V. Fedoryuk's survey, "Asymptotic methods in analysis"; however, some general theorems of this theory are presented in Chap. 7. The question of actually integrating particular equations is not touched on at all; the standard book on this subject is E. Kamke's "Differentialgleichungen: Lösungen und Lösungsmethoden", Chelsea (1948).

In recent years there has been a sharp increase in research activity involving classical problems of the theory of differential equations. This is due to the penetration into the theory of other disciplines: algebra (the theory of formal normal forms), algebraic geometry (resolution of singularities), and complex analysis. We have tried, as far as possible, to reflect this research in the present article.