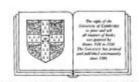
ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

BASIC HYPERGEOMETRIC SERIES

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