

Jean-Pierre Serre

Lectures on the Mordell-Weil Theorem

Translated and edited by Martin Brown
from notes by Michel Waldschmidt

3rd edition



CONTENTS

| | |
|---|----|
| 1. Summary. | 1 |
| 1.1. Heights. | 3 |
| 1.2. The Mordell-Weil theorem and Mordell's conjecture. | 3 |
| 1.3. Integral points on algebraic curves. Siegel's theorem. | 4 |
| 1.4. Baker's method. | 5 |
| 1.5. Hilbert's irreducibility theorem. Sieves. | 5 |
| 2. Heights. | 7 |
| 2.1. The product formula. | 7 |
| 2.2. Heights on $P_m(K)$. | 10 |
| 2.3. Properties of heights. | 13 |
| 2.4. Northcott's finiteness theorem. | 16 |
| 2.5. Quantitative form of Northcott's theorem. | 17 |
| 2.6. Height associated to a morphism $\phi: X \rightarrow P_n$. | 19 |
| 2.7. The group $\text{Pic}(X)$. | 20 |
| 2.8. Heights and line bundles. | 22 |
| 2.9. $h_c = O(1) \Leftrightarrow c$ is of finite order (number fields). | 24 |
| 2.10. Positivity of the height. | 24 |
| 2.11. Divisors algebraically equivalent to zero. | 25 |
| 2.12. Example-exercise: projective plane blown up at a point. | 26 |
| 3. Normalised heights. | 29 |
| 3.1. Néron-Tate normalisation. | 29 |
| 3.2. Abelian varieties. | 31 |
| 3.3. Quadraticity of \bar{h}_c on abelian varieties. | 35 |
| 3.4. Duality and Poincaré divisors. | 36 |
| 3.5. Example: elliptic curves. | 39 |
| 3.6. Exercises on elliptic curves. | 40 |
| 3.7. Applications to properties of heights. | 41 |
| 3.8. Non-degeneracy. | 42 |
| 3.9. Structure of $A(K)$: a preliminary result. | 43 |
| 3.10. Back to §2.11 (c algebraically equivalent to zero). | 44 |
| 3.11. Back to §2.9 (torsion c). | 46 |

| | |
|--|-----|
| 4. The Mordell-Weil theorem. | 49 |
| 4.1. Hermite's finiteness theorem. | 49 |
| 4.2. The Chevalley-Weil theorem. | 50 |
| 4.3. The Mordell-Weil theorem. | 51 |
| 4.4. The classical descent. | 53 |
| 4.5. The number of points of bounded height on an abelian variety. | 53 |
| 4.6. Explicit form of the weak Mordell-Weil theorem. | 55 |
| 5. Mordell's conjecture. | 58 |
| 5.1. Chabauty's theorem. | 58 |
| 5.2. The Manin-Demjanenko theorem. | 62 |
| 5.3. First application: Fermat quartics (Demjanenko). | 66 |
| 5.4. Second application: modular curves $X_0(p^n)$ (Manin). | 67 |
| 5.5. The generalised Mordell conjecture. | 73 |
| 5.6. Mumford's theorem; preliminaries. | 74 |
| 5.7. Application to heights: Mumford's inequality. | 77 |
| 6. Local calculation of normalised heights. | 81 |
| 6.1. Bounded sets. | 81 |
| 6.2. Local heights. | 83 |
| 6.3. Néron's theorem. | 87 |
| 6.4. Relation with global heights. | 89 |
| 6.5. Elliptic curves. | 90 |
| 7. Siegel's method. | 94 |
| 7.1. Quasi-integral sets. | 94 |
| 7.2. Approximation of real numbers. | 95 |
| 7.3. The approximation theorem on abelian varieties. | 98 |
| 7.4. Application to curves of genus ≥ 1 . | 101 |
| 7.5. Proof of Siegel's theorem. | 102 |
| 7.6. Application to $P(f(n))$. | 105 |
| 7.7. Effectivity. | 106 |
| 8. Baker's method. | 108 |
| 8.1. Reduction theorems. | 108 |
| 8.2. Lower bounds for $\sum \beta_i \log \alpha_i$. | 110 |
| 8.3. Application to $P_1 - \{0, 1, \infty\}$. | 112 |
| 8.4. Applications to other curves. | 114 |
| 8.5. Applications to elliptic curves with good reduction outside a given finite set of places. | 118 |

| | |
|---|-----|
| 9. Hilbert's irreducibility theorem. | 121 |
| 9.1. Thin sets. | 121 |
| 9.2. Specialisation of Galois groups. | 122 |
| 9.3. Examples of degrees 2,3,4,5. | 123 |
| 9.4. Further properties of thin sets. | 127 |
| 9.5. Hilbertian fields. | 129 |
| 9.6. The irreducibility theorem: elementary proof. | 130 |
| 9.7. Thin sets in P_1 : upper bounds. | 132 |
| 10. Construction of Galois extensions. | 137 |
| 10.1. The method. | 137 |
| 10.2. Extensions with Galois group S_n . | 138 |
| 10.3. Extensions with Galois group A_n . | 144 |
| 10.4. Further examples of Galois groups: use of elliptic curves. | 145 |
| 10.5. Noether's method. | 147 |
| 10.6. Infinite Galois extensions. | 147 |
| 10.7. Recent results. | 149 |
| 11. Construction of elliptic curves of large rank. | 152 |
| 11.1. Néron's specialisation theorem. | 152 |
| 11.2. Elliptic curves of rank ≥ 9 over \mathbb{Q} . | 154 |
| 11.3. Elliptic curves of rank ≥ 10 over \mathbb{Q} . | 158 |
| 11.4. Elliptic curves of rank ≥ 11 over \mathbb{Q} . | 161 |
| 12. The large sieve. | 163 |
| 12.1. Statement of the main theorem. | 163 |
| 12.2. A lemma on finite groups. | 164 |
| 12.3. The Davenport-Halberstam theorem. | 166 |
| 12.4. Proof of the Davenport-Halberstam theorem. | 167 |
| 12.5. End of the proof of the main theorem. | 172 |
| 13. Applications of the large sieve to thin sets. | 177 |
| 13.1. Statements of results. | 177 |
| 13.2. Proof of theorem 1. | 179 |
| 13.3. Proof of theorem 5. | 183 |
| 13.4. Proof of theorem 3 from theorem 1. | 186 |

| | |
|---|-----|
| Appendix: The class number 1 problem and integral points on modular curves. | 188 |
| A.1. Historical remarks. | 188 |
| A.2. Equivalent conditions for $h(-p) = 1$. | 190 |
| A.3. Orders of R_d . | 191 |
| A.4. Elliptic curves with complex multiplication. | 192 |
| A.5. Modular curves associated to normalisers of Cartan subgroups and their CM integral points. | 194 |
| A.6. Examples. | 196 |
| A.7. The Gel'fond-Linnik-Baker method. | 197 |
| Bibliography. | 200 |
| Index. | 210 |