Techniques in Fractal Geometry

Kenneth Falconer

University of St Andrews

JOHN WILEY & SONS

Chichester · New York · Weinheim · Brisbane · Singapore · Toronto

Contents

Preface ix

Introduction xi

Notes and references xvi

Chapter 1 Mathematical background 1

1.1 Sets and functions 1

1.2 Some useful inequalities 3

1.3 Measures 6

1.4 Weak convergence of measures 13

1.5 Notes and references 16

Exercises 16

Chapter 2 Review of fractal geometry 19

2.1 Review of dimensions 19

2.2 Review of iterated function systems 29

2.3 Notes and references 39

Exercises 39

Chapter 3 Some techniques for studying dimension 41

3.1 Implicit methods 41

3.2 Box-counting dimensions of cut-out sets 51

3.3 Notes and references 56

Exercises 57

Chapter 4 Cookie-cutters and bounded distortion 59

4.1 Cookie-cutter sets 59

4.2 Bounded distortion for cookie-cutters 62

4.3 Notes and references 69

Exercises 69

Chapter 5 The thermodynamic formalism 71

5.1 Pressure and Gibbs measures 71

5.2 The dimension formula 75

5.3 Invariant measures and the transfer operator 79

5.4 Entropy and the variational principle 84
5.5 Further applications 88
5.6 Why 'thermodynamic' formalism? 92
5.7 Notes and references 94
Exercises 95

Chapter 6 The ergodic theorem and fractals **97**
6.1 The ergodic theorem 97
6.2 Densities and average densities 102
6.3 Notes and references 111
Exercises 112

Chapter 7 The renewal theorem and fractals **113**
7.1 The renewal theorem 113
7.2 Applications to fractals 123
7.3 Notes and references 128
Exercises 128

Chapter 8 Martingales and fractals **129**
8.1 Martingales and the convergence theorem 129
8.2 A random cut-out set 136
8.3 Bi-Lipschitz equivalence of fractals 143
8.4 Notes and references 146
Exercises 146

Chapter 9 Tangent measures **149**
9.1 Definitions and basic properties 149
9.2 Tangent measures and densities 155
9.3 Singular integrals 163
9.4 Notes and references 167
Exercises 167

Chapter 10 Dimensions of measures **169**
10.1 Local dimensions and dimensions of measures 169
10.2 Dimension decomposition of measures 177
10.3 Notes and references 184
Exercises 184

Chapter 11 Some multifractal analysis **185**
11.1 Fine and coarse multifractal theories 186
11.2 Multifractal analysis of self-similar measures 192

11.3 Multifractal analysis of Gibbs measures on cookie-cutter sets 201
11.4 Notes and references 204
Exercises 205

Chapter 12 Fractals and differential equations **207**
12.1 The dimension of attractors 207
12.2 Eigenvalues of the Laplacian on regions with fractal boundary 223
12.3 The heat equation on regions with fractal boundary 230
12.4 Differential equations on fractal domains 236
12.5 Notes and references 244
Exercises 245

References **247**

Index **253**