

OPERATIONAL CALCULUS

BASED ON THE
TWO-SIDED LAPLACE INTEGRAL

BY

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'La langue de l'analyse, la plus parfaite de toutes les langues, étant par elle-même un puissant instrument de découvertes; ses notations, lorsqu'elles sont nécessaires et heureusement imaginées, sont les germes de nouveaux calculs.'

LAPLACE, *Théorie analytique des probabilités*,
Paris, 1814, p. 7

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CONTENTS

I	GENERAL INTRODUCTION	
1.	History of the operational calculus	page 1
2.	The operational calculus based on the Laplace transform	2
3.	Survey of the subject-matter	5
II	THE FOURIER INTEGRAL AS BASIS OF THE OPERATIONAL CALCULUS	
1.	The Fourier integral	7
2.	A pair of integrals equivalent to the Fourier identity	9
3.	Parallel displacement of the path of integration in the integral (5b)	11
4.	Rotation of the path of integration in the integral (11b)	12
5.	Strip of convergence of the Laplace integral	13
6.	The language of the operational calculus	18
7.	Operational calculus based on one-sided Laplace integrals	20
III	ELEMENTARY OPERATIONAL IMAGES	
1.	Introduction	22
2.	Images of polynomials; the Γ -function	23
3.	Images of simple exponential functions	26
4.	Images of complicated exponential functions	27
5.	Images of trigonometric functions	29
6.	Images of logarithmic functions	30
7.	Well-known integrals of the Laplace type	31
IV	ELEMENTARY RULES	
1.	Introduction	33
2.	The similarity rule	33
3.	The shift rule	34
4.	The attenuation rule	38
5.	The composition product	39
6.	Repeated composition product	43
7.	The image of the product of two originals	47
8.	The differentiation rule	48
9.	The integration rule	51
10.	Rules for multiplication by t^n	53
11.	Rules for division by t , and related integrals	53
12.	Rules for the treatment of correlation functions	55
V	THE DELTA OR IMPULSE FUNCTION	
1.	Introduction	56
2.	The unit function	56
3.	The δ -function as derivative of the unit function	59

4.	History of the impulse function	page 62
5.	The sifting integral as a Stieltjes integral	66
6.	The image of the δ -function	68
7.	Functions approximating the δ -function	70
8.	Applications of the δ -function	74
9.	Series of impulse functions	78
10.	Derivatives of the δ -function	80
11.	Note on the theory of distributions	84
VI QUESTIONS CONCERNING THE CONVERGENCE OF THE DEFINITION INTEGRAL		
1.	Introduction	85
2.	Extensions of the definition integral	85
3.	The operational treatment of some special series	87
4.	The strip of convergence of operational relations	91
5.	Particular cases of the strip of convergence	94
6.	Absolute convergence	96
7.	Uniform convergence	97
8.	Summable series	100
9.	Summable integrals	102
10.	The behaviour of the definition integral on the boundaries of the strip of convergence	104
11.	The inversion integral	106
12.	Adjacent strips of convergence for the same image	109
13.	Operational relations having a line of convergence	113
14.	Symmetry between the integrals of definition and inversion	115
15.	Uniqueness of the operational relations	117
VII ASYMPTOTIC RELATIONS AND OPERATIONAL TRANS- POSITION OF SERIES		
1.	Introduction	121
2.	Abel and Tauber theorems	122
3.	Abel theorems	124
4.	Real Tauber theorems	128
5.	Complex Tauber theorems	130
6.	Operational equalities	133
7.	Asymptotic series	134
8.	Operational transposition of power series in p^{-1} and in t	136
9.	Transposition of series with ascending powers of p	139
10.	Expansion in rational fractions (Heaviside's expansion theorem II)	142
11.	Transposition of other series	147
12.	A real inversion formula	148

VIII LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

1. Introduction	page 152
2. Inhomogeneous equations with the unit function at the right	152
3. Inhomogeneous equations with arbitrary right-hand member	155
4. Differential equations with boundary conditions	157
5. Admittances and impedances	161
6. Transient phenomena	167
7. Time impedances	170
8. Series for small values of t . Heaviside's first expansion theorem	171
9. Classification of admittances	173

IX SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS; ELECTRIC-CIRCUIT THEORY

1. Introduction	175
2. Equations of electric-circuit theory	175
3. Transposition of the circuit equations; mutual impedances and admittances	177
4. Time admittances of electric circuits	180
5. Applications of the general circuit theory	182
6. Circuit equations with initial conditions	184
7. Ladder networks; filters	185
8. Lattice networks	194

X LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

1. Introduction	200
2. Laguerre polynomials	201
3. Hermite polynomials	204
4. Bessel functions (operational relations)	207
5. Bessel functions (applications)	214
6. Legendre functions (operational relations)	220
7. Legendre functions (applications)	224
8. Hypergeometric functions	225

XI OPERATIONAL RULES OF MORE COMPLICATED CHARACTER

1. Introduction	232
2. Rules obtained when p is replaced by a function of p	232
3. Rules obtained when t is replaced by a function of t	237
4. The exponential transformation: $t \rightarrow e^t$	238
5. Exponential operational relations for power series, in particular hypergeometric series	243
6. Rules concerning series expansions	249

7.	Equalities in connexion with a single operational relation	page 253
8.	Equalities in connexion with a pair of simultaneous relations (exchange identity)	254
XII STEP FUNCTIONS AND OTHER DISCONTINUOUS FUNCTIONS		
1.	Introduction	257
2.	Operational relations involving step functions jumping at integral values of t	258
3.	The saw-tooth function	260
4.	Arithmetic functions in connexion with θ -functions	263
5.	Arithmetic functions in connexion with Dirichlet series	266
6.	Step functions of argument equal to the summation variable in a series	272
7.	Contragrade series	274
XIII DIFFERENCE EQUATIONS		
1.	Introduction	277
2.	Difference equation for the 'sum'	278
3.	General linear difference equations with constant coefficients	281
4.	Linear difference equations with variable coefficients	285
5.	Connexion between differential and difference equations	287
6.	Operational construction of difference equations	289
XIV INTEGRAL EQUATIONS		
1.	Introduction	292
2.	The integral equation for the moving average	293
3.	Integral equations of the first kind with difference kernel	300
4.	Integral equations of the first kind with kernel reducible to a difference kernel	305
5.	Integral equations of the second kind with a difference kernel	307
6.	Homogeneous integral equations	310
7.	The operational construction of integral equations	312
8.	Note on the operational interpretation of the Wiener-Hopf technique	313
XV PARTIAL DIFFERENTIAL EQUATIONS IN THE OPERATIONAL CALCULUS OF ONE VARIABLE		
1.	Introduction	314
2.	Homogeneous linear partial differential equations (general solutions)	317
3.	Homogeneous partial differential equations with boundary conditions	322
4.	Quasi-stationary theory of electric cables	325
5.	Inhomogeneous partial differential equations	330

XVI SIMULTANEOUS OPERATIONAL CALCULUS

1. Introduction	page 334
2. General theory	334
3. Second-order differential equations of the hyperbolic type with constant coefficients and two variables	344
4. Hyperbolic differential equations of the second order in more than two independent variables	352
5. Elliptic differential equations	355
6. Simultaneous partial differential equations	361
7. Partial difference equations	365

XVII 'GRAMMAR'	373
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XVIII 'DICTIONARY'	383
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LIST OF AUTHORS QUOTED	411
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GENERAL INDEX	413
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